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MATH 172 Exam 1 Spring 2020

Sections 501 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Compute $\int_0^{\pi/2} \cos^{3/2}x \sin x\,dx$.

a. $\frac{1}{5}$

b. $\frac{2}{5}$ correct choice

c. $\frac{4}{5}$

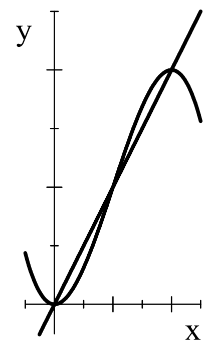
d. $\frac{5}{2}$

e. 5

Solution: $u = \cos x \qquad du = -\sin x\,dx$.

$$\int_0^{\pi/2} \cos^{3/2}x \sin x\,dx = -\int_1^0 u^{3/2}\,du = -\left[\frac{2u^{5/2}}{5}\right]_1^0 = 0 - -\frac{2}{5} = \frac{2}{5}$$

2. Find the total area between $y = 3x^2 - x^3$ and $y = 2x$.



a. 0

b. $\frac{1}{4}$

c. $\frac{1}{2}$ correct choice

d. 1

e. 2

Solution: The curves intersect when $3x^2 - x^3 = 2x$ or

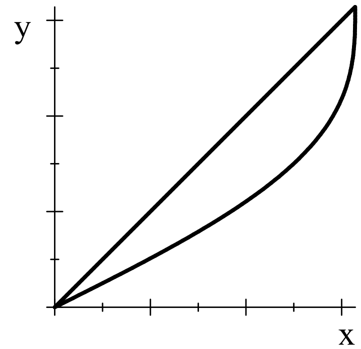
$0 = x^3 - 3x^2 + 2x = x(x^2 - 3x + 2) = x(x-1)(x-2)$ or $x = 0, 1, 2$. So

$$A = \int_0^1 (2x - 3x^2 + x^3)\,dx + \int_1^2 (3x^2 - x^3 - 2x)\,dx = \left[x^2 - x^3 + \frac{x^4}{4}\right]_0^1 + \left[x^3 - \frac{x^4}{4} - x^2\right]_1^2$$

$$= \left(1 - 1 + \frac{1}{4}\right) + (8 - 4 - 4) - \left(1 - \frac{1}{4} - 1\right) = \frac{1}{2}$$

1-14	/56	16	/18
15	/17	17	/15
		Total	/106

3. Find the area between $x = y + \sin y$ and $x = y$.



a. π

b. $\frac{\pi}{2}$

c. 4

d. 3

e. 2 correct choice

Solution: The curves intersect when $y + \sin y = y$ or $\sin y = 0$ or $y = 0, \pi$.

$$A = \int_0^\pi (y + \sin y) - (y)\,dy = [-\cos y]_0^\pi = (--1) - (-1) = 2$$

4. Find the area between $y = 3x\sqrt{16 + x^2}$ and the x-axis for $0 \le x \le 3$

a. 61 correct choice

b. 9

c. $3^{3/2}$

d. 54

e. 244

Solution: $A = \int_0^3 3x\sqrt{16 + x^2}\,dx$ We substitute $u = 16 + x^2$ and $du = 2x\,dx$ and $\frac{1}{2}\,du = x\,dx$.

$$A = \frac{3}{2}\int_{16}^{25} \sqrt{u}\,du = \left[u^{3/2}\right]_{16}^{25} = 125 - 64 = 61$$

5. Compute $\int_0^1 2xe^{2x}\,dx$.

a. $\frac{1}{2}(e^2 - 1)$

b. $\frac{1}{2}(e^2 + 1)$ correct choice

c. $\frac{1}{2}e^2$

d. $\frac{1}{2}(3e^2 - 1)$

e. $\frac{3}{2}e^2$

Solution: Use parts with

$$u = x \qquad dv = 2e^{2x}\,dx$$
$$du = dx \qquad v = e^{2x}$$

$$I = xe^{2x} - \int e^{2x}\,dx = \left[xe^{2x} - \frac{1}{2}e^{2x}\right]_0^1 = \left(e^2 - \frac{1}{2}e^2\right) - \left(-\frac{1}{2}\right) = \frac{1}{2}e^2 + \frac{1}{2}$$

6. Find the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

a. $\frac{2}{\pi}$ correct choice
b. $\frac{1}{\pi}$
c. 2π
d. π
e. 2

Solution: $f_{\text{ave}} = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}\Big[-\cos x\Big]_0^\pi = \frac{1}{\pi}(--1--1) = \frac{2}{\pi}$

7. Find the length of the parametric curve $x = \theta$ and $y = \ln(\cos\theta)$ for $0 \le \theta \le \frac{\pi}{4}$.

a. $\ln\left|\sqrt{2}+1\right|+1$
b. $\ln\left|\sqrt{2}-1\right|+1$
c. $\ln\left|\sqrt{2}+1\right|-1$
d. $\ln\left|\sqrt{2}-1\right|$
e. $\ln\left|\sqrt{2}+1\right|$ correct choice

Solution: $\frac{dx}{d\theta} = 1 \qquad \frac{dy}{d\theta} = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$

$$L = \int_0^{\pi/4}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_0^{\pi/4}\sqrt{1+\tan^2\theta}\,d\theta = \int_0^{\pi/4}\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4}$$
$$= \ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right| - \ln|\sec 0+\tan 0| = \ln\left|\sqrt{2}+1\right| - \ln|1+0| = \ln\left|\sqrt{2}+1\right|$$

8. The curve $(x,y) = \left(t+1, \frac{t^2}{2}+t\right)$ for $0 \le t \le 1$ is rotated about the y-axis. Find the surface area.

a. $\frac{2\pi}{3}$
b. $\frac{8\pi}{3}5^{3/2}$
c. $\frac{2\pi}{3}(5^{3/2}-2^{3/2})$ correct choice
d. $\frac{8\pi}{3}(5^{3/2}-1)$
e. $\frac{8\pi}{3}(5^{3/2}-2^{3/2})$

Solution: $\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = t+1$. The radius is $r = x = t+1$. So the surface area is:

$$A = \int 2\pi r\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 2\pi(t+1)\sqrt{1+(t+1)^2}\,dt = \pi\int_0^1(2t+2)\sqrt{t^2+2t+2}\,dt$$

Let $u = t^2+2t+2$. Then $du = (2t+2)\,dt$ So

$$A = \pi\int_2^5\sqrt{u}\,du = \pi\frac{2u^{3/2}}{3}\Big|_2^5 = \frac{2\pi}{3}(5^{3/2}-2^{3/2})$$

9. Compute $\int_1^2 x^3 \ln x\,dx$.

a. $2\ln 2 - \frac{7}{8}$

b. $2\ln 2 - \frac{9}{8}$

c. $2\ln 2 - \frac{15}{16}$

d. $4\ln 2 - \frac{15}{16}$ correct choice

e. $4\ln 2 - \frac{17}{16}$

Solution: Parts with $\begin{array}{ll} u = \ln x & dv = x^3\,dx \\ du = \frac{1}{x}\,dx & v = \frac{1}{4}x^4 \end{array}$. Then

$$I = \left[\frac{1}{4}x^4 \ln x - \frac{1}{4}\int x^4 \frac{1}{x}\,dx\right]_1^2 = \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4\right]_1^2 = (4\ln 2 - 1) - \left(\frac{1}{4}\ln 1 - \frac{1}{16}\right) = 4\ln 2 - \frac{15}{16}$$

10. Compute $\int_{\pi/4}^{\pi/3} \tan^3\theta \sec^2\theta\,d\theta$.

a. 2 correct choice

b. 4

c. 20

d. $\frac{1}{4}$

e. $\frac{1}{2}$

Solution: Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$. So

$$\int_{\pi/4}^{\pi/3} \tan^3\theta \sec^2\theta\,d\theta = \int_1^{\sqrt{3}} u^3\,du = \left[\frac{u^4}{4}\right]_1^{\sqrt{3}} = \left(\frac{9}{4} - \frac{1}{4}\right) = 2$$

11. Compute $\int_0^\pi \cos^4\theta\,d\theta$.

a. $\frac{3\pi}{16}$

b. $\frac{5\pi}{16}$

c. $\frac{5\pi}{8}$

d. $\frac{3\pi}{8}$ correct choice

e. $\frac{3\pi}{4}$

Solution: Use $\cos^2 A = \frac{1+\cos 2A}{2}$.

$$\int_0^\pi \cos^4\theta\,d\theta = \int_0^\pi \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta = \frac{1}{4}\int_0^\pi (1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta = \frac{1}{4}\int_0^\pi \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \frac{1}{4}\left[\theta + \sin 2\theta + \frac{1}{2}\left(\theta + \frac{\sin 4\theta}{4}\right)\right]_0^\pi = \frac{1}{4}\left(\pi + \frac{\pi}{2}\right) = \frac{3\pi}{8}$$

12. Compute $\int \frac{1}{(4-x^2)^{3/2}}\,dx$

a. $\frac{1}{4\sqrt{4-x^2}} + C$

b. $\frac{x}{4\sqrt{4-x^2}} + C$ correct choice

c. $\frac{1}{2\sqrt{4-x^2}} + C$

d. $\frac{\sqrt{4-x^2}}{4x} + C$

e. $\frac{\sqrt{4-x^2}}{2x} + C$

Solution: Let $x = 2\sin\theta$. Then $dx = 2\cos\theta\,d\theta$. So

$$I = \int \frac{1}{(4-x^2)^{3/2}}\,dx = \int \frac{1}{(4-4\sin^2\theta)^{3/2}}\,2\cos\theta\,d\theta = \frac{1}{4}\int \frac{1}{(\cos^2\theta)^{3/2}}\,\cos\theta\,d\theta = \frac{1}{4}\int \frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{1}{4}\tan\theta + C$$

Draw a triangle with opposite side x, hypotenuse 2 and adjacent side $\sqrt{4-x^2}$. So

$$I = \frac{x}{4\sqrt{4-x^2}} + C$$

13. Compute $\int 9x^2\cos(3x)\,dx$.

a. $3x^2\sin(3x) + 2x\cos(3x) + \frac{2}{3}\sin(3x) + C$

b. $3x^2\sin(3x) - 2x\cos(3x) - \frac{2}{3}\sin(3x) + C$

c. $3x^2\sin(3x) + 2x\cos(3x) - \frac{2}{3}\sin(3x) + C$ correct choice

d. $3x^2\sin(3x) - 2x\cos(3x) + \frac{2}{3}\sin(3x) + C$

Solution: Use parts with $u = x^2$, $dv = 9\cos(3x)\,dx$, $du = 2x\,dx$, $v = 3\sin(3x)$

$$I = 3x^2\sin(3x) - \int 6x\sin(3x)\,dx$$

Now use parts with $u = x$, $dv = 6\sin(3x)\,dx$, $du = dx$, $v = -2\cos(3x)$

$$I = 3x^2\sin(3x) - \left[-2x\cos(3x) + 2\int \cos(3x)\,dx\right]$$

$$I = 3x^2\sin(3x) + 2x\cos(3x) - \frac{2}{3}\sin(3x) + C$$

14. A rocket takes off from rest ($v = 0$) at the ground ($y = 0$) and has acceleration $a = 40e^{-2t}$.
Find its height at $t = 2$.

a. $10e^{-4}$

b. $40e^{-4}$

c. $160e^{-4}$

d. $10e^{-4} + 30$ correct choice

e. $10e^{-4} + 10$

Solution: $\frac{dv}{dt} = a = 40e^{-2t}$ $\quad v = -20e^{-2t} + C$ $\quad v(0) = -20 + C = 0$ $\quad \Rightarrow \quad C = 20$

$\frac{dy}{dt} = v = -20e^{-2t} + 20$ $\quad y = 10e^{-2t} + 20t + K$ $\quad y(0) = 10 + K = 0$ $\quad \Rightarrow \quad K = -10$

$y = 10e^{-2t} + 20t - 10$ $\quad y(2) = 10e^{-2\cdot 2} + 20 \cdot 2 - 10 = 30 + 10e^{-4}$

Work Out: (Points indicated. Part credit possible. Show all work.)

15. (17 points) A bar between $x = 1$ and $x = 9$ has linear density $\delta = \frac{1}{\sqrt{x}}$ g/cm.

a. Find the total mass of the bar.

Solution: $M = \int \delta\, dx = \int_1^9 \frac{1}{\sqrt{x}}\, dx = \left[2\sqrt{x}\right]_1^9 = 2(3 - 1) = 4$

b. Find the center of mass of the bar.

Solution: $M_1 = \int x\delta\, dx = \int_1^9 \frac{x}{\sqrt{x}}\, dx = \int_1^9 \sqrt{x}\, dx = \left[\frac{2x^{3/2}}{3}\right]_1^9 = \frac{2}{3}(27 - 1) = \frac{52}{3}$

$\bar{x} = \frac{M_1}{M} = \frac{52}{3}\frac{1}{4} = \frac{13}{3}$

16. (18 points) Compute $\int e^{4x} \cos 3x \, dx$.

Solution: Use parts with $\begin{array}{ll} u = \cos 3x & dv = e^{4x}\, dx \\ du = -3\sin 3x\, dx & v = \frac{1}{4} e^{4x} \end{array}$. Then

$$I = \int e^{4x} \cos 3x \, dx = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{4} \int e^{4x} \sin 3x \, dx$$

Next use parts with $\begin{array}{ll} u = \sin 3x & dv = e^{4x}\, dx \\ du = 3\cos 3x\, dx & v = \frac{1}{4} e^{4x} \end{array}$

$$I = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{4}\left[\frac{1}{4} e^{4x} \sin 3x - \frac{3}{4} \int e^{4x} \cos 3x \, dx\right] = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{16} e^{4x} \sin 3x - \frac{9}{16} I$$

$$I + \frac{9}{16} I = \frac{1}{4} e^{4x} \cos 3x + \frac{3}{16} e^{4x} \sin 3x$$

$$I = \frac{16}{25}\left(\frac{1}{4} e^{4x} \cos 3x + \frac{3}{16} e^{4x} \sin 3x\right) + C = \frac{4}{25} e^{4x} \cos 3x + \frac{3}{25} e^{4x} \sin 3x + C$$

17. (15 points) Compute $\int \frac{1}{x^2 \sqrt{x^2 - 4}} \, dx$.

Solution: Let $x = 2\sec\theta$. Then $dx = 2\sec\theta \tan\theta \, d\theta$. So

$$I = \int \frac{1}{x^2 \sqrt{x^2 - 4}} \, dx = \int \frac{1}{4\sec^2\theta \sqrt{4\sec^2\theta - 4}} 2\sec\theta \tan\theta \, d\theta = \frac{1}{4} \int \frac{1}{\sec\theta} \, d\theta = \frac{1}{4} \int \cos\theta \, d\theta$$

$$= \frac{1}{4} \sin\theta + C$$

Draw a triangle with hypotenous x, adjacent side 2 and opposite side $\sqrt{x^2 - 4}$. Then

$$I = \frac{1}{4} \frac{\sqrt{x^2 - 4}}{x} + C$$