Name____________________________________

1-12	/60	14	/15
13	/15	15	/15
		Total	/105

MATH 172 Final Spring 2021

Sections 501 Solutions P. Yasskin

Anything above 100 is extra credit.

Multiple Choice and Short Answer: (5 Points Each)

1. Compute $\int_1^e 3x^2 \ln x\, dx$.

a. $\frac{2}{3}e^3 - \frac{1}{3}$ d. $\frac{2}{3}e^3$ g. $\frac{2}{3}e^3 + \frac{1}{3}$ correct choice

b. $\frac{4}{3}e^3 - \frac{1}{3}$ e. $\frac{4}{3}e^3$ h. $\frac{4}{3}e^3 + \frac{1}{3}$

c. $2e^3 - \frac{1}{3}$ f. $2e^3$ i. $2e^3 + 1$

Solution: Integrate by parts $u = \ln x \quad dv = 3x^2\, dx$, $du = \frac{1}{x}\, dx \quad v = x^3$

$$\int_1^e 3x^2 \ln x\, dx = \Big[x^3 \ln x\Big]_1^e - \int_1^e x^2\, dx = \left[x^3 \ln x - \frac{x^3}{3}\right]_1^e = \left[e^3 - \frac{e^3}{3}\right] - \left[-\frac{1}{3}\right] = \frac{2}{3}e^3 + \frac{1}{3}$$

2. Compute $\int_0^{\pi/4} \sin^2 x \cos^2 x\, dx$.

a. $\frac{\pi}{32}$ correct choice

b. $\frac{\pi}{16}$

c. $\frac{\pi}{8}$

d. $\frac{\pi}{4}$

e. $\frac{\pi}{2}$

Solution: $\sin(2x) = 2\sin x \cos x$ So $\sin^2 x \cos^2 x = \frac{1}{4}\sin^2(2x) = \frac{1}{4}\frac{1-\cos(4x)}{2}$

$$\int_0^{\pi/4} \sin^2 x \cos^2 x\, dx = \frac{1}{8}\int_0^{\pi/4} 1 - \cos(4x)\, dx = \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right]_0^{\pi/4} = \frac{1}{8}\frac{\pi}{4} = \frac{\pi}{32}$$

3. Compute $\int \frac{1}{(x^2+4)^{3/2}}\,dx$

a. $\frac{\sqrt{x^2-4}}{4x}+C$

b. $\frac{x}{4\sqrt{x^2+4}}+C$ correct choice

c. $\frac{1}{2}\arctan\frac{x}{2}+\frac{x}{4\sqrt{x^2+4}}+C$

d. $\frac{1}{4}\arctan\frac{x}{4}+\frac{\sqrt{x^2-4}}{4x}+C$

e. $\frac{x}{2}\arctan\frac{x}{2}+C$

f. $\frac{x}{4}\arctan\frac{x}{2}+C$

Solution: $x = 2\tan\theta \qquad dx = 2\sec^2\theta\,d\theta$

$$\int \frac{1}{(x^2+4)^{3/2}}\,dx = \int \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta+4)^{3/2}} = \frac{1}{4}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^{3/2}} = \frac{1}{4}\int \cos\theta\,d\theta = \frac{1}{4}\sin\theta + C$$

Since $\tan\theta = \frac{x}{2}$, draw a triangle with opposite side x and adjacent side 2.
Then the hypotenuse is $\sqrt{x^2+4}$ and so $\sin\theta = \frac{x}{\sqrt{x^2+4}}$. Therefore:

$$\int \frac{1}{(x^2+4)^{3/2}}\,dx = \frac{1}{4}\frac{x}{\sqrt{x^2+4}}+C$$

Check: $$\frac{d}{dx}\frac{1}{4}\frac{x}{\sqrt{x^2+4}} = \frac{1}{4}\frac{\sqrt{x^2+4}-x\frac{x}{\sqrt{x^2+4}}}{x^2+4} = \frac{1}{4}\frac{(x^2+4)-x^2}{(x^2+4)\sqrt{x^2+4}} = \frac{1}{(x^2+4)^{3/2}}$$

4. Find the area between the line $y = x$ and the parabola $x = 5y - y^2$.

a. 36
b. $\frac{80}{3}$
c. $\frac{32}{3}$ correct choice
d. 18
e. $\frac{25}{3}$

Solution: We do a y integral. To find the limits, we equate $5y-y^2 = y \qquad 4y-y^2=0 \qquad y = 0,4$
To see which is bigger, we plug in $y=2$. $\quad x = y = 2 \qquad x = 5y-y^2 = 10-4 = 6$

$$A = \int_0^4 (5y-y^2-y)\,dy = \int_0^4 (4y-y^2)\,dy = \left[2y^2 - \frac{y^3}{3}\right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$$

5. Find the average value of the function $f(x) = 6x - x^2$ on $[0,6]$.

a. 180
b. 36
c. 30
d. 6 correct choice
e. $\frac{9}{2}$

Solution: $$f_{\text{ave}} = \frac{1}{6}\int_0^6 (6x-x^2)\,dx = \frac{1}{6}\left[3x^2 - \frac{x^3}{3}\right]_0^6 = \frac{1}{6}(3\cdot 36 - 2\cdot 36) = 6$$

6. Find the center of mass of a 2 m bar whose density is $\delta = \dfrac{1}{x^3}$ for $2 \le x \le 4$.

a. $\dfrac{7}{3}$
b. $\dfrac{1}{4}$
c. $\dfrac{3}{8}$
d. $\dfrac{8}{3}$ correct choice
e. $\dfrac{5}{2}$

Solution: $M = \int_2^4 \delta\,dx = \int_2^4 \dfrac{1}{x^3}\,dx = \left[\dfrac{-1}{2x^2}\right]_2^4 = -\dfrac{1}{32} + \dfrac{1}{8} = \dfrac{4-1}{32} = \dfrac{3}{32}$

$M_1 = \int_2^4 x\delta\,dx = \int_2^4 \dfrac{1}{x^2}\,dx = \left[\dfrac{-1}{x}\right]_2^4 = -\dfrac{1}{4} + \dfrac{1}{2} = \dfrac{1}{4}$ $\quad \bar{x} = \dfrac{M_1}{M} = \dfrac{1}{4}\dfrac{32}{3} = \dfrac{8}{3}$

7. Find the arc length of the parametric curve $\vec{r}(t) = \left(\frac{1}{2}t^2, \frac{1}{3}t^3\right)$ for $0 \le t \le \sqrt{3}$.

a. 3
b. $\dfrac{8}{3}$
c. $\dfrac{7}{3}$ correct choice
d. 2
e. $\dfrac{4}{3}$

Solution: $\dfrac{dx}{dt} = t \qquad \dfrac{dy}{dt} = t^2$

$L = \int_0^{\sqrt{3}} \sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}\,dt = \int_0^{\sqrt{3}} \sqrt{(t)^2 + (t^2)^2}\,dt = \int_0^{\sqrt{3}} t\sqrt{1+t^2}\,dt = \left[\dfrac{(1+t^2)^{3/2}}{3}\right]_0^{\sqrt{3}}$

$= \dfrac{(1+3)^{3/2}}{3} - \dfrac{(1)^{3/2}}{3} = \dfrac{8}{3} - \dfrac{1}{3} = \dfrac{7}{3}$

8. The region between the parabola $x = 6y - y^2$ and the y-axis is rotated about the x-axis. Find the volume swept out.

a. $V = 2 \cdot 6^2\pi$
b. $V = 3 \cdot 6^2\pi$
c. $V = 6^3\pi$ correct choice
d. $V = \dfrac{6^4}{5}\pi$
e. $V = 5 \cdot 6^4\pi$

Solution: We do a y-integral. The rectangles are horizontal. A rectangle sweeps out a cylinder. The radius is y. The limits are $y = 0, 6$. So the volume is:

$V = \int_0^6 2\pi r h\,dy = \int_0^6 2\pi y(6y - y^2)\,dy = 2\pi\left[2y^3 - \dfrac{y^4}{4}\right]_0^6 = 2\pi\left(2 \cdot 6^3 - \dfrac{6^4}{4}\right) = 6^3\pi$

9. Find the area inside the spiral $r = e^{\theta}$ for $0 \le \theta \le \pi$.

a.	$\frac{1}{4}(e^{2\pi} - 1)$	correct choice	i.	$\frac{1}{4}e^{2\pi}$	q.	$\frac{1}{4}(e^{2\pi} + 1)$
b.	$\frac{1}{2}(e^{2\pi} - 1)$		j.	$\frac{1}{2}e^{2\pi}$	r.	$\frac{1}{2}(e^{2\pi} + 1)$
c.	$e^{2\pi} - 1$		k.	$e^{2\pi}$	s.	$e^{2\pi} + 1$
d.	$2(e^{2\pi} - 1)$		l.	$2e^{2\pi}$	t.	$2(e^{2\pi} + 1)$
e.	$\frac{1}{4}(e^{\pi} - 1)$		m.	$\frac{1}{4}e^{\pi}$	u.	$\frac{1}{4}(e^{\pi} + 1)$
f.	$\frac{1}{2}(e^{\pi} - 1)$		n.	$\frac{1}{2}e^{\pi}$	v.	$\frac{1}{2}(e^{\pi} + 1)$
g.	$e^{\pi} - 1$		o.	e^{π}	w.	$e^{\pi} + 1$
h.	$2(e^{\pi} - 1)$		p.	$2e^{\pi}$	x.	$2(e^{\pi} + 1)$

Solution: $A = \int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} \frac{1}{2} e^{2\theta} d\theta = \left[\frac{1}{4} e^{2\theta} \right]_0^{\pi} = \frac{1}{4} e^{2\pi} - \frac{1}{4} e^{0} = \frac{1}{4}(e^{2\pi} - 1)$

10. Find the arc length of the spiral $r = e^{\theta}$ for $0 \le \theta \le \pi$.

a.	$e^{2\pi} + 1$	g.	$e^{2\pi}$	m.	$e^{2\pi} - 1$	
b.	$\sqrt{2}\,(e^{2\pi} + 1)$	h.	$\sqrt{2}\,e^{2\pi}$	n.	$\sqrt{2}\,(e^{2\pi} - 1)$	
c.	$2(e^{2\pi} + 1)$	i.	$2e^{2\pi}$	k.	$2(e^{2\pi} - 1)$	
d.	$e^{\pi} + 1$	j.	e^{π}	o.	$e^{\pi} + 1$	
e.	$\sqrt{2}\,(e^{\pi} + 1)$	k.	$\sqrt{2}\,e^{\pi}$	p.	$\sqrt{2}\,(e^{\pi} - 1)$	correct choice
f.	$2(e^{\pi} + 1)$	l.	$2e^{\pi}$	q.	$2(e^{\pi} - 1)$	

Solution: $L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta = \int_0^{\pi} \sqrt{2e^{2\theta}}\, d\theta$

$= \int_0^{\pi} \sqrt{2}\, e^{\theta} d\theta = \left[\sqrt{2}\, e^{\theta} \right]_0^{\pi} = \sqrt{2}\,(e^{\pi} - e^{0}) = \sqrt{2}\,(e^{\pi} - 1)$

11. Find the Taylor series for $f(x) = \frac{1}{x}$ about $x = 2$.

a. $\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$	e. $\sum_{n=0}^{\infty} \frac{n!}{2^n} x^n$	i. $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$
b. $\sum_{n=0}^{\infty} \frac{1}{2^n} (x-2)^n$	f. $\sum_{n=0}^{\infty} \frac{n!}{2^n} (x-2)^n$	j. $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (x-2)^n$
c. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n$	g. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} x^n$	k. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$
d. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-2)^n$	h. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} (x-2)^n$	l. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$ correct choice

Solution: We make a table of the function and several derivatives and evaluate at $x = 2$. We then generalize to the n^{th} derivative:

$$f(x) = \frac{1}{x} \qquad f(2) = \frac{1}{2}$$
$$f'(x) = -\frac{1}{x^2} \qquad f'(2) = -\frac{1}{2^2}$$
$$f''(x) = \frac{2}{x^3} \qquad f''(2) = \frac{2}{2^3}$$
$$f'''(x) = -\frac{3!}{x^4} \qquad f'''(2) = -\frac{3!}{2^4}$$
$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \qquad f^{(n)}(2) = (-1)^n \frac{n!}{2^{n+1}}$$

Finally, we plug into the Taylor series:

$$Tf = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{n!}{2^{n+1}}}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$$

12. Compute $\lim_{n\to\infty} n^2\left[1 - \cos\left(\frac{1}{n}\right)\right]$.

a. -1	d. 0	g. 1
b. $-\frac{1}{2}$	e. $\frac{1}{4}$	h. π
c. $-\frac{1}{4}$	f. $\frac{1}{2}$ correct choice	i. 2π

Solution: As $n \to \infty$, we have $\frac{1}{n} \to 0$, and $\cos\frac{1}{n} \to 1$, and $1 - \cos\left(\frac{1}{n}\right) \to 0$.
So the limit has the indeterminate form $\infty \cdot 0$. Let $t = \frac{1}{n}$. Then

$$\lim_{n\to\infty} n^2\left[1 - \cos\left(\frac{1}{n}\right)\right] = \lim_{t\to 0^+} \frac{1-\cos(t)}{t^2} \overset{l'H}{=} \lim_{t\to 0^+} \frac{\sin(t)}{2t} \overset{l'H}{=} \lim_{t\to 0^+} \frac{\cos(t)}{2} = \frac{1}{2}$$

Or using the Maclaurin series

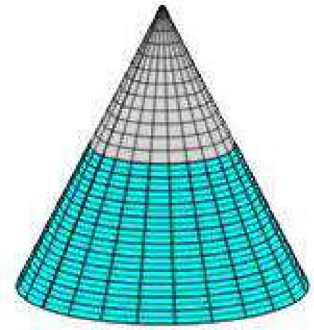
$$\lim_{n\to\infty} n^2\left[1 - \cos\left(\frac{1}{n}\right)\right] = \lim_{t\to 0^+} \frac{1-\cos(t)}{t^2} = \lim_{t\to 0^+} \frac{1 - \left(1 - \frac{t^2}{2} + \cdots\right)}{t^2} = \frac{1}{2}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (15 points) Work Out Problem

A water tank has the shape of a cone with the vertex at the top. Its height is $H = 16$ ft and its radius is $R = 8$ ft. It is filled with salt water to a depth of 10 ft which weighs $\delta = 64\ \frac{\text{lb}}{\text{ft}^3}$.

Find the work done to pump the water out the top of the tank.



Solution: Put the y-axis measuring down from the top.

The slice which is a distance y down from the top is a circle of radius r.

By similar triangles, $\frac{r}{y} = \frac{R}{H} = \frac{8}{16} = \frac{1}{2}$. So $r = \frac{1}{2}y$.

The area is $A = \pi r^2 = \frac{\pi y^2}{4}$ and the volume of the slice of thickness dy is $dV = A\,dy = \frac{\pi y^2}{4}\,dy$.

It weighs $dF = \delta\,dV = 64\frac{\pi y^2}{4}\,dy = 16\pi y^2\,dy$. It is lifted a distance $D = y$.

There is water between $y = 6$ and $y = 16$. So the work done is

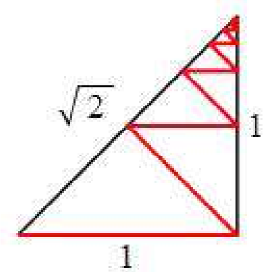
$$W = \int_6^{16} D\,dF = \int_6^{16} y\,16\pi y^2\,dy = \left[16\pi\frac{y^4}{4}\right]_6^{16} = 4\pi(16^4 - 6^4)\text{ ft-lb}$$

14. (15 points) Work Out Problem

Find the length of the infinite zigzag within the $45°$ right triangle, shown at the right.

Each diagonal is at $45°$.

The total length includes the base.



$L =$ __$2 + \sqrt{2}$__

Solution: Each horizontal line has half the length of the previous and starts with 1.

Each diagonal line has half the length of the previous and starts with $\frac{\sqrt{2}}{2}$.

So the total length is

$$L = \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) + \frac{\sqrt{2}}{2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \left(1 + \frac{\sqrt{2}}{2}\right)\frac{1}{1 - \frac{1}{2}} = 2 + \sqrt{2}$$

15. (15 points) Work Out Problem

Find the interval of convergence of the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+1} \frac{(x-4)^n}{2^n}$.

a. Find the radius of convergence.

Solution: We apply the Ratio Test:

$$|a_n| = \frac{1}{\sqrt{n}+1} \frac{|x-4|^n}{2^n} \qquad |a_{n+1}| = \frac{1}{\sqrt{n+1}+1} \frac{|x-4|^{n+1}}{2^{n+1}}$$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+1} \frac{|x-4|^{n+1}}{2^{n+1}} \frac{\sqrt{n}+1}{1} \frac{2^n}{|x-4|^n}$$

$$= \frac{|x-4|}{2} \lim_{n\to\infty} \frac{\sqrt{n}+1}{\sqrt{n+1}+1} = \frac{|x-4|}{2} \lim_{n\to\infty} \frac{1-n^{-1/2}}{1-(n+1)^{-1/2}} = \frac{|x-4|}{2} < 1$$

$|x-4| < 2 \qquad R = 2 \qquad$ Open interval: $\quad (2,6)$

b. Check the convergence at the left endpoint.
Be sure to name any convergence test you use and check out all conditions.

Solution: $x = 2$: $\quad \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+1} \frac{(-2)^n}{2^n} = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n}+1}$

We compare to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series since $p = \frac{1}{2} < 1$. We compute the limit:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n}+1} = \lim_{n\to\infty} \frac{1}{1-n^{-1/2}} = 1$$ Since $0 < L < \infty$

by the Limit Comparison Test, the original series also diverges.

c. Check the convergence at the right endpoint.
Be sure to name any convergence test you use and check out all conditions.

Solution: $x = 6$: $\quad \displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+1} \frac{(2)^n}{2^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}+1}$

This converges by the Alternating Series Test, because $b_n = \dfrac{1}{\sqrt{n}+1}$ is positive, decreasing and $\displaystyle\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}+1} = 0$.

d. State the interval of convergence.

Solution: The interval of convergence is $\quad (2,6]$.