Name________________________________

MATH 172	Exam 2	Spring 2023
Sections 502	Solutions	P. Yasskin

1-11	/55	13	/16
12	/16	14	/18
		Total	/105

Multiple Choice: (5 points each. No part credit. Circle your answers.)

1. Find the general partial fraction expansion of $f(x) = \dfrac{(x+2)^2}{(x^4-16)(x-2)}$.

a. $\dfrac{A}{(x-2)^2} + \dfrac{Bx+C}{x^2+4}$

b. $\dfrac{A}{x-2} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{x^2+4}$ Correct

c. $\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4}$

d. $\dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{x^2-4}$

e. $\dfrac{A}{(x-2)^2} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2}$

Solution: We factor $x^4-16 = (x^2+4)(x+2)(x-2)$ and cancel the $(x+2)$. Then there are linear denominators for $(x-2)$ and $(x-2)^2$ and a quadradic denominator for (x^2+4). The linears get a constant on top and the quadradic gets a linear on top:

$$f(x) = \frac{(x+2)^2}{(x^2+4)(x+2)(x-2)^2} = \frac{(x+2)}{(x^2+4)(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+4}$$

2. Given the partial fraction expansion:

$$\frac{x^2+32x-4}{x^4-16} = \frac{2}{x-2} + \frac{2}{x+2} + \frac{-4x+1}{x^2+4}$$

which term in the following integral is INCORRECT?

$$\int \frac{x^2+32x-4}{x^4-16}dx = \underbrace{\ln|x-2|^2}_{\text{A}} + \underbrace{\ln|x+2|^2}_{\text{B}} - \underbrace{\ln|x^2+4|^2}_{\text{C}} + \underbrace{\frac{1}{2}\arctan\left(\frac{x}{2}\right)}_{\text{D}}$$

a. A

b. B

c. C

d. D

e. They are all correct. Correct

Solution: Simplfy using a log identity and then differentiate:

$$\frac{d}{dx}\ln|x-2|^2 = \frac{d}{dx}2\ln|x-2| = \frac{2}{x-2} \qquad \frac{d}{dx}\ln|x+2|^2 = \frac{d}{dx}2\ln|x+2| = \frac{2}{x+2}$$

$$\frac{d}{dx} - \ln|x^2+4|^2 = \frac{d}{dx} - 2\ln|x^2+4| = \frac{-2(2x)}{x^2+4} = \frac{-4x}{x^2+4}$$

$$\frac{d}{dx}\frac{1}{2}\arctan\left(\frac{x}{2}\right) = \frac{1}{2}\frac{1}{1+\left(\frac{x}{2}\right)^2}\frac{1}{2} = \frac{1}{4+x^2}$$

They are all correct.

3. $\int \frac{1}{(x^2-9)^{3/2}}\,dx =$

a. $\frac{1}{3}\frac{1}{\sqrt{x^2-9}}$

b. $\frac{1}{3}\frac{x}{\sqrt{x^2-9}}$

c. $\frac{1}{9}\frac{1}{\sqrt{x^2-9}}$

d. $\frac{1}{9}\frac{x}{\sqrt{x^2-9}}$

e. $-\frac{1}{9}\frac{x}{\sqrt{x^2-9}}$ Correct

Solution: We use a $\sec$ substitution, because $x > 3$. So we substitute $x = 3\sec\theta$ and $dx = 3\sec\theta\tan\theta\,d\theta$:

$$\int \frac{1}{(x^2-9)^{3/2}}\,dx = \int \frac{1}{(9\sec^2\theta-9)^{3/2}} 3\sec\theta\tan\theta\,d\theta = \frac{1}{9}\int \frac{1}{(\tan^2\theta)^{3/2}}\sec\theta\tan\theta\,d\theta = \frac{1}{9}\int \frac{1}{\tan^2\theta}\sec\theta\,d\theta$$

$$= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{9}\int \frac{1}{u^2}\,du = -\frac{1}{9}\frac{1}{u} = -\frac{1}{9}\frac{1}{\sin\theta} = -\frac{1}{9}\frac{x}{\sqrt{x^2-9}}$$ (Draw a triangle.)

4. $\int_0^4 \frac{1}{(9+x^2)^{3/2}}\,dx =$

a. $\frac{1}{15}$

b. $\frac{1}{45}$

c. $\frac{4}{45}$ Correct

d. $\frac{4}{135}$

e. $\frac{4}{225}$

Solution: We use a $\tan$ substitution, because of the plus sign. So we substitute $x = 3\tan\theta$ and $dx = 3\sec^2\theta\,d\theta$:

$$\int_0^4 \frac{1}{(9+x^2)^{3/2}}\,dx = \int \frac{1}{(9+9\tan^2\theta)^{3/2}} 3\sec^2\theta\,d\theta = \frac{1}{9}\int \frac{1}{(\sec^2\theta)^{3/2}}\sec^2\theta\,d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta = \frac{1}{9}\frac{x}{\sqrt{9+x^2}}\Bigg|_0^4$$ (Draw a triangle.)

$$= \frac{1}{9}\frac{4}{\sqrt{9+16}} = \frac{4}{45}$$

5. $\int_0^4 \frac{1}{x^2 - 25}\,dx =$

a. $-\frac{1}{5}\ln 3$ Correct

b. $\frac{1}{5}\ln 3$

c. $\frac{1}{5}\ln 4 - \frac{1}{5}$

d. $-\frac{1}{5}\ln 4 + \frac{1}{5}$

e. $\frac{1}{5}\ln 4$

Solution: We use a $\sin$ substitution, because the limits say $x < 5$. So we substitute $x = 5\sin\theta$ and $dx = 5\cos\theta\,d\theta$:

$$\int_0^4 \frac{1}{x^2 - 25}\,dx = \int \frac{1}{25\sin^2\theta - 25} 5\cos\theta\,d\theta = -\frac{1}{5}\int \frac{1}{\cos\theta}\,d\theta = -\frac{1}{5}\int \sec\theta\,d\theta = -\frac{1}{5}\ln|\sec\theta + \tan\theta|$$

$$= \left[-\frac{1}{5}\ln\left|\frac{5}{\sqrt{25 - x^2}} + \frac{x}{\sqrt{25 - x^2}}\right|\right]_0^4 \quad \text{(Draw a triangle.)}$$

$$= -\frac{1}{5}\ln\left|\frac{5}{\sqrt{9}} + \frac{4}{\sqrt{9}}\right| + \frac{1}{5}\ln\left|\frac{5}{\sqrt{25}}\right| = -\frac{1}{5}\ln\frac{9}{\sqrt{9}} = -\frac{1}{5}\ln 3$$

Alternatively, we could use the partial fraction expansion $\frac{1}{x^2 - 25} = \frac{1}{10(x-5)} - \frac{1}{10(x+5)}$.

6. Consider the integrals:

$$A = \int_3^4 \frac{1}{(x-3)^{2/3}}\,dx \qquad B = \int_3^4 \frac{1}{(x-3)^{4/3}}\,dx \qquad C = \int_4^\infty \frac{1}{(x-3)^{2/3}}\,dx \qquad D = \int_4^\infty \frac{1}{(x-3)^{4/3}}\,dx$$

Which are finite? Which are infinite?

a. A and B are finite. C and D are infinite.

b. B and C are finite. A and D are infinite.

c. B and D are finite. A and C are infinite.

d. A and D are finite. B and C are infinite. Correct

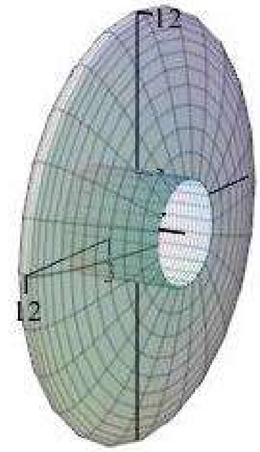
e. A and C are finite. B and D are infinite.

Solution: For large x, notice $\frac{1}{(x-3)^{4/3}}$ is more damped than $\frac{1}{x-3}$. So D is finite.

For large x, notice $\frac{1}{(x-3)^{2/3}}$ is less damped than $\frac{1}{x-3}$. So C is infinite.

Near $x = 3$, the behavior is reversed. So B is infinite and A is finite.

7. The region between $y = 12 - x^2$ and $y = 3$ is rotated about the x-axis.
Which integral gives the volume swept out?



a. $V = \pi \int_{-3}^{3} (x^4 - 24x^2 + 135)\, dx$ Correct

b. $V = 2\pi \int_{-3}^{3} (x^4 - 24x^2 + 135)\, dx$

c. $V = \pi \int_{0}^{3} (9x - x^3)\, dx$

d. $V = 2\pi \int_{0}^{3} (9x - x^3)\, dx$

e. $V = 2\pi \int_{-3}^{3} (9x - x^3)\, dx$

Solution: Each y is a function of x. So we do an x-integral. They intersect when

$$12 - x^2 = 3 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$$

The slices are vertical and rotate about the x-axis into washers.
The inner radius is $r = 3$ and the outer radius is $R = 12 - x^2$.
So the volume is

$$V = \int_{-3}^{3} \pi R^2 - \pi r^2\, dx = \pi \int_{-3}^{3} (12 - x^2)^2 - (3)^2\, dx = \pi \int_{-3}^{3} (x^4 - 24x^2 + 135)\, dx$$

8. The region between $y = 12 - x^2$ and $y = 3$ is rotated about the y-axis.
Find the volume swept out.

a. $\frac{81\pi}{4}$

b. $\frac{81\pi}{2}$ Correct

c. 18π

d. 36π

e. 81π

Solution: Each y is a function of x. So we do an x-integral.
The slices are vertical and rotate about the y-axis into cylinders.
The radius is $r = x$ and the height is $h = (12 - x^2) - 3 = 9 - x^2$.
The functions intersect when

$$12 - x^2 = 3 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3$$

However, if we integrate from $x = -3$ to $x = 3$, we are double counting the volume.
So we integrate from $x = 0$ to $x = 3$. So the volume is

$$V = \int_{0}^{3} 2\pi r h\, dx = 2\pi \int_{0}^{3} x(9 - x^2)\, dx = 2\pi \int_{0}^{3} (9x - x^3)\, dx = 2\pi \left[9\frac{x^2}{2} - \frac{x^4}{4} \right]_0^3 = \frac{81\pi}{2}$$

9. The base of a solid is the region between $y = x^2$ and the x-axis for $0 \le x \le 3$. The cross sections perpendicular to the x-axis are squares. Find the volume of the solid.

a. 9

b. 27

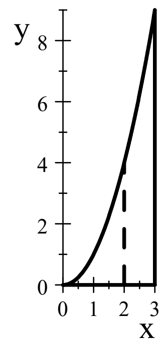
c. 81

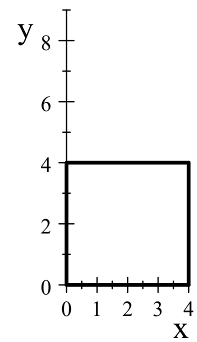
d. $\frac{3^5}{5}$ Correct

e. $\frac{3^4}{4}$

Solution: Here are plots of the base, with a slice perpendicular to the x-axis and a cross section. The area of the cross section is $A = y^2 = (x^2)^2 = x^4$. So the volume is

$$V = \int_0^3 A\,dx = \int_0^3 x^4\,dx = \left[\frac{x^5}{5}\right]_0^3 = \frac{3^5}{5}$$





10. A spring has a rest length of $x_0 = 5$ m. It requires 12 N of force to hold the spring at $x = 7$ m. Find the work done to stretch the spring from $x = 6$ m to $x = 8$ m.

a. 6

b. 8

c. 12

d. 18

e. 24 Correct

Solution: $F = k(x - x_0)$ $\quad 12 = k(7 - 5)$ $\quad k = 6$ $\quad F = 6(x - 5)$

$$W = \int F\,dx = \int_6^8 6(x - 5)\,dx = 3(x - 5)^2\Big|_6^8 = 3(3)^2 - 3(1)^2 = 24$$

11. A 20 ft rope hangs from the top of a building. It's linear weight density is $\rho = 3$ lb/ft. How much work is done to lift the rope to the top of the building?

a. 600 ft-lb Correct

b. 450 ft-lb

c. 300 ft-lb

d. 200 ft-lb

e. 150 ft-lb

Solution: The piece of rope of length dy which is y ft from the top of the building is lifted $D = y$ ft and has weight $dF = 3\,dy$. So the work is

$$W = \int D\,dF = \int_0^{20} y 3\,dy = 3\frac{y^2}{2}\Big|_0^{20} = 3\frac{20^2}{2} = 600 \text{ ft-lb}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. $(16$ points$)$ Find the coefficients in the partial fraction expansion

$$\frac{10}{(x^2+4)(x^2-1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1} + \frac{D}{x-1}$$

Solution: Clear the denominator:

$$10 = (Ax+B)(x^2-1) + C(x^2+4)(x-1) + D(x^2+4)(x+1)$$

Plug in $x=1$: $10 = (A+B)(0) + C(1+4)(0) + D(1+4)(1+1) = 10D$

$\Rightarrow \quad D = 1$

Plug in $x=-1$: $10 = (-A+B)(0) + C(1+4)(-2) + D(1+4)(0) = -10C$

$\Rightarrow \quad C = -1$

Plug in $x=0$: $10 = B(-1) + C(4)(-1) + D(4)(1) = -B - 4C + 4D = -B + 8$

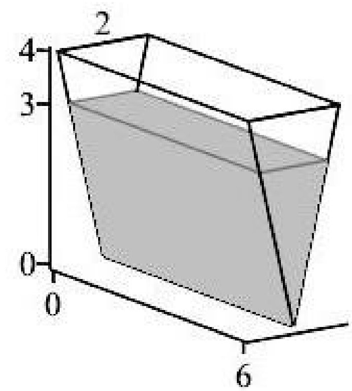
$\Rightarrow \quad B = -2$

Coeff of x^3: $0 = A + C + D = A - 1 + 1 = A$

$\Rightarrow \quad A = 0$

$A = 0$
$B = -2$
$C = -1$
$D = 1$

13. $(16$ points$)$ The tank shown is 6 m long, 2 m wide at the top and 4 m high. It is filled with water to a depth of 3 m.
How much work is done to pump the water out the top of the tank?
Take the density of water to be ρ kg/m^3 and the acceleration of gravity to be g m/sec^2. (You don't need numbers for ρ and g.)



Solution: Put the 0 of the y-axis at the bottom of the tank and measure y upward.
The slice at height y has to be lifted a distance $D = 4 - y$.
The slice at height y is a rectangle of length 6 and width w, and so area $A = 6w$.
By similar triangles $\frac{w}{y} = \frac{2}{4} = \frac{1}{2}$. So $w = \frac{y}{2}$ and $A = 6\frac{y}{2} = 3y$.
The slice at height y with thickness dy has volume $dV = A\,dy = 3y\,dy$ and weight $dF = \rho g\,dV = 3\rho g y\,dy$.
There is water for $0 \le y \le 3$. (This is the tricky part.) So the work is:

$$W = \int_0^3 D\,dF = \int_0^3 (4-y)\,3\rho g y\,dy = \rho g\left[6y^2 - y^3\right]_0^3 = \rho g(54-27) = 27\rho g$$

14. $\left(18 \text{ points}\right)$ Consider the integral $\int_1^9 (x-4)^2\,dx$ The exact value is $\frac{152}{3}$.
Use each of the following numerical techniques to approximate the integral.

a. Left Riemann Sum with 4 intervals

Solution: $\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = 2.$ The evaluation points are $x_i = 1, 3, 5, 7.$
The function values are $f(x_i) = 9, 1, 1, 9.$ The Left Riemann Sum is
$L_4 = (9+1+1+9)2 = 40$

b. Right Riemann Sum with 4 intervals

Solution: $\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = 2.$ The evaluation points are $x_i = 3, 5, 7, 9.$
The function values are $f(x_i) = 1, 1, 9, 25.$ The Right Riemann Sum is
$R_4 = (1+1+9+25)2 = 72$

c. Midpoint Riemann Sum with 4 intervals

Solution: $\Delta x = \frac{b-a}{n} = \frac{9-1}{4} = 2.$ The evaluation points are $x_i = 2, 4, 6, 8.$
The function values are $f(x_i) = 4, 0, 4, 16.$ The Right Riemann Sum is
$M_4 = (4+0+4+16)2 = 48$

d. Trapezoid Rule with 4 intervals

Solution: $T_4 = \frac{L_4 + R_4}{2} = \frac{40+72}{2} = 56$
Alternatively:
$T_4 = \left(\frac{1}{2}9 + 1 + 1 + 9 + \frac{1}{2}25\right)2 = 56$

e. Simpson's Rule with 4 intervals

Solution: For quadratic functions, Simpson's Rule is exact. So $S_4 = \frac{152}{3}$.
Alternatively:
$S_4 = \frac{1}{3}(9 + 4 \cdot 1 + 2 \cdot 1 + 4 \cdot 9 + 25)2 = \frac{152}{3}$