Name____

MATH 172H Exam 3 Spring 2019

Sections 200 Solutions P. Yasskin

15 Multiple Choice: (4 points each. No part credit.)

1. Compute $\displaystyle\lim_{n\to\infty} \frac{(-2)^n - (-3)^n}{(-3)^n}$.

a. -2
b. -1 correct choice
c. 1
d. 2
e. diverges

Solution: $\displaystyle\lim_{n\to\infty} \frac{(-2)^n - (-3)^n}{(-3)^n} \cdot \frac{\frac{1}{(-3)^n}}{\frac{1}{(-3)^n}} = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n - 1}{1} = -1$

2. Compute $\displaystyle\lim_{n\to\infty} \left(\sqrt{n^4 + 4n^2} - \sqrt{n^4 - 2n^2}\right)$.

a. $-\infty$
b. -6
c. 3 correct choice
d. 6
e. ∞

Solution: $\displaystyle\lim_{n\to\infty} \left(\sqrt{n^4 + 4n^2} - \sqrt{n^4 - 2n^2}\right) \frac{\sqrt{n^4 + 4n^2} + \sqrt{n^4 - 2n^2}}{\sqrt{n^4 + 4n^2} + \sqrt{n^4 - 2n^2}} = \lim_{n\to\infty} \frac{(n^4 + 4n^2) - (n^4 - 2n^2)}{\sqrt{n^4 + 4n^2} + \sqrt{n^4 - 2n^2}}$

$\displaystyle= \lim_{n\to\infty} \frac{6n^2}{\sqrt{n^4 + 4n^2} + \sqrt{n^4 - 2n^2}} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{6}{\sqrt{1 + 4n^{-2}} + \sqrt{1 - 2n^{-2}}} = \frac{6}{2} = 3$

1-15	/60	17	/15
16	/10	18	/20
		Total	/105

3. Compute $\lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)^n$

a. 0
b. e^{-4}
c. e^{-2}
d. e^{-1}
e. 1 correct choice

Solution: $\lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)^n = e^L$ where

$$L = \lim_{n\to\infty} n\ln\left(1-\frac{2}{n^2}\right) = \lim_{n\to\infty}\frac{\ln\left(1-\frac{2}{n^2}\right)}{\frac{1}{n}} \overset{\text{l'H}}{=} \lim_{n\to\infty}\frac{\dfrac{\frac{4}{n^3}}{1-\frac{2}{n^2}}}{\frac{-1}{n^2}} = \lim_{n\to\infty}\frac{\frac{4}{n^3}}{1-\frac{2}{n^2}}(-n^2) = \lim_{n\to\infty}\frac{\frac{-4}{n}}{1-\frac{2}{n^2}} = 0$$

So $\lim_{n\to\infty}\left(1-\frac{2}{n^2}\right)^n = e^L = e^0 = 1$

4. If $S = \sum_{n=1}^{\infty} a_n$ and $S_k = \frac{k}{k+1}$, then

a. $a_n = \frac{-1}{n(n+1)}$
b. $a_n = \frac{1}{n(n-1)}$
c. $a_n = \frac{2}{n(n-1)}$
d. $a_n = \frac{1}{n(n+1)}$ correct choice
e. $a_n = \frac{2}{n(n+1)}$

Solution: $a_n = S_n - S_{n-1} = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$

5. Compute $\sum_{n=2}^{\infty}\left(\frac{2n+1}{n} - \frac{2n+3}{n+1}\right)$

a. $\frac{1}{2}$ correct choice
b. $\frac{3}{2}$
c. 2
d. -2
e. 0

Solution: Telescoping

$$S_k = \sum_{n=2}^{k}\left(\frac{2n+1}{n} - \frac{2n+3}{n+1}\right) = \left(\frac{5}{2}-\frac{7}{3}\right) + \left(\frac{7}{3}-\frac{9}{4}\right) + \cdots + \left(\frac{2k+1}{k} - \frac{2k+3}{k+1}\right) = \frac{5}{2} - \frac{2k+3}{k+1}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\frac{5}{2} - \frac{2k+3}{k+1}\right) = \frac{5}{2} - 2 = \frac{1}{2}$$

6. For this and the next problem, consider the series $\sum_{n=0}^{\infty} \frac{1}{e^n + 1}$. This series

a. converges to a number less than e^{-1}
b. converges to a number less than $\frac{e}{e-1}$ correct choice
c. converges to a number greater than $\frac{e}{e-1}$
d. diverges to ∞
e. diverges but not to ∞

Solution: Compare to $\sum_{n0}^{\infty} \frac{1}{e^n} = \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n$ which is a geometric series which converges because $\frac{1}{e} < 1$. Then $\sum_{n=0}^{\infty} \frac{1}{e^n + 1} < \sum_{n=0}^{\infty} \frac{1}{e^n} = \frac{1}{1 - e^{-1}} = \frac{e}{e-1}$

7. Which test did you use in the previous problem?

a. Integral Test
b. Simple Comparison Test correct choice
c. Limit (but not Simple) Comparison Test
d. Alternating Series Test
e. n^{th} Term Divergence Test

Solution: $\frac{1}{e^n + 1} < \frac{1}{e^n}$ So we only needed the Simple Comparison Test.

8. The series $\sum_{n=1}^{\infty} \frac{2n+2}{n^2 + 2n}$

a. converges by the Integral Test
b. diverges by the Integral Test correct choice
c. converges by a Simple Comparison with $\sum_{n=1}^{\infty} \frac{2}{n^2}$
d. diverges by a Simple Comparison with $\sum_{n=1}^{\infty} \frac{2}{n}$
e. converges by the Ratio Test

Solution: $\frac{2n+2}{n^2 + 2n}$ is positive, decreasing and continuous.

$\int_1^{\infty} \frac{2n+2}{n^2 + 2n}\, dn = \left[\ln(n^2 + 2n)\right]_1^{\infty} = \infty$ So $\sum_{n=1}^{\infty} \frac{2n+2}{n^2 + 2n}$ diverges by the Integral Test.

9. The series $S = \sum_{n=1}^{\infty} \frac{2n+2}{(n^2+2n)^2}$ converges by the Integral Test. If we approximate S by $S_{10} = \sum_{n=1}^{10} \frac{2n+2}{(n^2+2n)^2}$, find a bound on the error $E_{10} = S - S_{10} = \sum_{n=11}^{\infty} \frac{2n+2}{(n^2+2n)^2}$.

a. $|E_{10}| < \frac{1}{120}$ correct choice
b. $|E_{10}| < \frac{1}{143}$
c. $|E_{10}| < \frac{1}{150}$
d. $|E_{10}| < \frac{1}{160}$
e. $|E_{10}| < \frac{1}{180}$

Solution: $|E_{10}| < \int_{10}^{\infty} \frac{2n+2}{(n^2+2n)^2}\, dn = \left[-\frac{1}{n^2+2n}\right]_{10}^{\infty} = 0 - -\frac{1}{100+20} = \frac{1}{120}$

10. For this and the next problem, consider the series $\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$. This series

a. converges correct choice
b. diverges to ∞
c. diverges to $-\infty$
d. diverges but not to $\pm\infty$

Solution: For $n \geq 2$, $n^2 > \sqrt{n}$. Compare to $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which converges because it is a p-series with $p = 2 > 1$. Since $\frac{1}{n^2 - \sqrt{n}} > \frac{1}{n^2}$, we cannot use the Simple Comparison test. We use the Limit Comparison Test.

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^2 - \sqrt{n}} \frac{n^2}{1} = 1$$

Since $0 < 1 < \infty$, $\sum_{n=2}^{\infty} \frac{1}{n^2 - \sqrt{n}}$ also converges.

11. Which test did you use in the previous problem?

a. Integral Test
b. Simple Comparison Test
c. Limit Comparison Test but not the Simple Comparison Test correct choice
d. Alternating Series Test
e. n^{th} Term Divergence Test

Solution: See the previous solution.

12. The series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$ is

a. absolutely convergent
b. conditionally convergent correct choice
c. divergent
d. conditionally divergent

Solution: The series is convergent by the Alternating Series Test because $\frac{1}{n+\sqrt{n}}$ is positive, decreasing and $\lim_{n\to\infty} \frac{1}{n+\sqrt{n}} = 0$. The related absolute series is $\sum_{n=2}^{\infty} \frac{1}{n+\sqrt{n}}$ which is divergent by the Limit Comparison Test with the harmonic series $\sum_{n=2}^{\infty} \frac{1}{n}$.

13. The series $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ is convergent by

a. the Alternating Series Test
b. the Related Absolute Series Test, the Simple Comparison Test and the p-Series Test correct choice
c. the Related Absolute Series Test, the Limit (but not Simple) Comparison Test and the p-Series Test
d. the n^{th} Term Divergence Test

Solution: The series is not alternating because $\cos n$ does not alternate. The related absolute series is $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ which we compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series because $p = 2 > 1$. Since $\frac{|\cos n|}{n^2} < \frac{1}{n^2}$, the series $\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ converges by the Simple Comparison Test and $\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ by the Related Absolute Series Test.

Note: The Limit Comparison Test will not work because $\lim_{n\to\infty} |\cos n|$ does not exist.

14. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{3n+2}{(-4)^n}(x-2)^n$

a. $R = \infty$
b. $R = 3$
c. $R = 4$ correct choice
d. $R = \frac{1}{3}$
e. $R = \frac{1}{4}$

Solution: We apply the Ratio Test. $|a_n| = \frac{3n+2}{4^n}|x-2|^n$ $\quad |a_{n+1}| = \frac{3n+5}{4^{n+1}}|x-2|^{n+1}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(3n+5)|x-2|^{n+1}}{4^{n+1}} \frac{4^n}{(3n+2)|x-2|^n} = \frac{|x-2|}{4} \lim_{n\to\infty} \frac{3n+5}{3n+2} = \frac{|x-2|}{4} < 1$$

$|x-2| < 4$ So $R = 4$.

15. Find the radius of convergence of the series $\sum_{n=1}^{\infty} \frac{(2n+1)!}{3^n}(x-5)^n$

a. $R = \infty$
b. $R = 3$
c. $R = 5$
d. $R = \frac{1}{3}$
e. $R = 0$ correct choice

Solution: We apply the Ratio Test. $|a_n| = \frac{(2n+1)!|x-5|^n}{3^n}$ $\quad |a_{n+1}| = \frac{(2n+3)!|x-5|^{n+1}}{3^{n+1}}$

$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(2n+3)!|x-5|^{n+1}}{3^{n+1}} \frac{3^n}{(2n+1)!|x-5|^{nn}} = \frac{|x-5|}{3} \lim_{n\to\infty} \frac{(2n+3)!}{(2n+1)!}$
$= 3|x-5| \lim_{n\to\infty}(2n+3)(2n+2) = \infty > 1$ for all x. So $R = 0$.

Work Out: (Points indicated. Part credit possible. Show all work.)

16. (10 points) Prove $\lim_{n\to\infty} \frac{1}{n^3} = 0.$

a. Write out the $\varepsilon - N$ definition of this limit.

Solution: For all $\varepsilon > 0$, there is an $N > 0$, such that if $n > N$ then $\left|\frac{1}{n^3}\right| < \varepsilon.$

b. Given an ε, what N should you use?

Solution: We work backwards: $\frac{1}{n^3} < \varepsilon \quad \Leftarrow \quad n^3 > \varepsilon \quad \Leftarrow \quad n > \sqrt[3]{\varepsilon}$
So given ε, we should take $N = \sqrt[3]{\varepsilon}$.

c. Complete the proof.

Solution: Given ε, let $N = \sqrt[3]{\varepsilon}$. Then if $n > N = \sqrt[3]{\varepsilon}$, then $n^3 > \varepsilon$ and $\frac{1}{n^3} < \varepsilon.$

17. (15 points) Determine whether the recursively defined sequence $a_1 = 4$ and $a_{n+1} = 3\sqrt{a_n}$ is convergent or divergent. If convergent, find the limit.

a. Find the first 3 terms: $a_1 =$ ______ $a_2 =$ ______ $a_3 =$ ______

Solution: $a_1 =$ ___4___ $a_2 =$ ___6___ $a_3 =$ ___$3\sqrt{6}$___

b. Assuming the limit $\lim\limits_{n\to\infty} a_n$ exists, find the possible limits.

Solution: Assume $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$ also. From the recursion relation:

$$L = 3\sqrt{L} \qquad L^2 = 9L \qquad L^2 - 9L = 0 \qquad L = 0,9$$

c. Prove the sequence is increasing or decreasing (as appropriate).

Solution: From the first 3 terms, we expect the sequence is increasing. So we want to prove $a_{n+1} > a_n > 0$.
Initialization Step: $a_2 = 6 > a_1 = 4 > 0$
Induction Step: Assume $a_{k+1} > a_k > 0$. We need to prove $a_{k+2} > a_{k+1} > 0$.
Proof:

$$a_{k+1} > a_k > 0 \quad \Rightarrow \quad \sqrt{a_{k+1}} > \sqrt{a_k} > 0 \quad \Rightarrow \quad 3\sqrt{a_{k+1}} > 3\sqrt{a_k} > 0 \quad \Rightarrow \quad a_{k+2} > a_{k+1} > 0$$

d. Prove the sequence is bounded or unbounded above or below (as appropriate).

Solution: From the possible limits, we expect the sequence is bounded above by 9. So we want to prove $a_n < 9$.
Initialization Step: $a_1 = 4 < 9$
Induction Step: Assume $a_k < 9$. We need to prove $a_{k+1} < 9$.
Proof:

$$a_k < 9 \quad \Rightarrow \quad \sqrt{a_k} < 3 \quad \Rightarrow \quad 3\sqrt{a_k} < 9 \quad \Rightarrow \quad a_{k+1} < 9$$

e. State whether the sequence is convergent or divergent and name the theorem. If convergent, state the limit.

Solution: The sequence is convergent by the Bounded Monotonic Sequence Theorem and $\lim\limits_{n\to\infty} a_n = 9$.

18. (20 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+1)3^n}(x-5)^n$.

a. Find the radius of convergence and state the open interval of absolute convergence.

$R =$ ____. Absolutely convergent on (_____,_____).

Solution: To find the radius, we use the Ratio Test. $|a_n| = \frac{\sqrt{n}\,|x-5|^n}{(n+1)3^n}$ $|a_{n+1}| = \frac{\sqrt{n+1}\,|x-5|^{n+1}}{(n+2)3^{n+1}}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\sqrt{n+1}\,|x-5|^{n+1}}{(n+2)3^{n+1}} \frac{(n+1)3^n}{\sqrt{n}\,|x-5|^n} = \frac{|x-5|}{3} \lim_{n\to\infty} \frac{n+1}{n+2}\sqrt{\frac{n+1}{n}} = \frac{|x-5|}{3} > 1$$

$|x-5| < 3$ So $R = 3$. Absolutely convergent on $(2,8)$

b. Check the **Left** Endpoint:

$x =$_____ The series is_________________________ Circle one:

Reasons: $\boxed{\text{Convergent}}$

Divergent

Solution: $x = 2$: $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+1)3^n}(-3)^n = \sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{(n+1)}$$

This converges by the Alternating Series Test because $\frac{\sqrt{n}}{(n+1)}$ is positive, decreasing and $\lim_{n\to\infty} \frac{\sqrt{n}}{(n+1)} = 0$.

c. Check the **Right** Endpoint:

$x =$_____ The series is_________________________ Circle one:

Reasons: Convergent

$\boxed{\text{Divergent}}$

Solution: $x = 8$: $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+1)3^n}(3)^n = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$$

Compare this to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2} < 1$ and so diverges.

We can't use the Simple Comparison Test because $\frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$. So we compute:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

Since $0 < 1 < \infty$, the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+1}$ diverges by the Limit Comparison Test.

d. State the Interval of Convergence. Interval= ____________

Solution: The Interval of Convergence.is: $[2,8)$