Name________________________________

MATH 172H	Exam 2	Fall 2019
Sections 202	Solutions	P. Yasskin

1-12	/60	11	/22
10	/22	Total	/104

12 Multiple Choice: (5 points each. No part credit.)

1. Find the arclength of the parametric curve $(x,y) = (2t^2, t^3)$ between $t = 0$ and $t = 1$.

a. $\frac{1}{27}$

b. $\frac{37}{27}$

c. $\frac{61}{27}$ correct choice

d. $\frac{1}{27}(13^{3/2} - 8)$

e. $\frac{1}{27}(13^{1/2} - 4)$

Solution: $ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(4t)^2 + (3t^2)^2}\, dt = t\sqrt{16 + 9t^2}\, dt$

$L = \int_0^1 ds = \int_0^1 t\sqrt{16 + 9t^2}\, dt = \left[\frac{1}{27}(16 + 9t^2)^{3/2}\right]_0^1 = \frac{125 - 64}{27} = \frac{61}{27}$

2. If the parametric curve $(x,y) = (2t^2, t^3)$ between $t = 0$ and $t = 1$ is rotated about the y-axis, set up the integral for the area of the surface swept out.

a. $\int_0^1 4\pi t^2 \sqrt{4 + 9t^2}\, dt$

b. $\int_0^1 2\pi t^4 \sqrt{16 + 9t^2}\, dt$

c. $\int_0^1 4\pi t^4 \sqrt{16 + 9t^2}\, dt$

d. $\int_0^1 2\pi t^3 \sqrt{4 + 9t^2}\, dt$

e. $\int_0^1 4\pi t^3 \sqrt{16 + 9t^2}\, dt$ correct choice

Solution: $ds = \sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(4t)^2 + (3t^2)^2}\, dt = t\sqrt{16 + 9t^2}\, dt$

Since we rotate about the y-axis, the radius is $r = x = 2t^2$.

$A = \int_0^1 2\pi r\, ds = \int_0^1 2\pi 2t^2 t\sqrt{16 + 9t^2}\, dt = \int_0^1 4\pi t^3 \sqrt{16 + 9t^2}\, dt$

3. Consider the partial fraction expansion

$$\frac{2}{(x-2)(x-3)^2(x-4)} = \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-4)}$$

What is $A+B+C+D$?

a. -2 correct choice

b. 2

c. -3

d. 3

e. -4

Solution: Clear the denominator.

$$2 = A(x-3)^2(x-4) + B(x-2)(x-3)(x-4) + C(x-2)(x-4) + D(x-2)(x-3)^2$$

Plug in $x=2$, $x=3$ and $x=4$.

$x=2$: $2 = A(-1)^2(-2)$ $\Rightarrow$ $A=-1$

$x=3$: $2 = C(1)(-1)$ $\Rightarrow$ $C=-2$

$x=4$: $2 = D(2)(1)^2$ $\Rightarrow$ $D=1$

Plug in $x=0$ and use A, C and D.

$x=0$: $2 = A(-3)^2(-4) + B(-2)(-3)(-4) + C(-2)(-4) + D(-2)(-3)^2$

$= 36 - 24B - 16 - 18 = 2 - 24B$ $\Rightarrow$ $B=0$

Then $A+B+C+D = -1+0-2+1 = -2$

4. Given the general partial fraction expansion $\dfrac{4(x+1)}{x^2(x+2)^2} = \dfrac{1}{x^2} - \dfrac{1}{(x+2)^2}$,

find $\displaystyle\int_2^4 \frac{4(x+1)}{x^2(x+2)^2}\,dx$.

a. $\frac{1}{3}$

b. $\frac{1}{4}$

c. $\frac{1}{6}$ correct choice

d. $\frac{1}{12}$

e. $\frac{11}{72}$

Solution: $\displaystyle\int_2^4 \frac{4(x+1)}{x^2(x+2)^2}\,dx = \int_2^4 \frac{1}{x^2} - \frac{1}{(x+2)^2}\,dx = \left[\frac{-1}{x} + \frac{1}{x+2}\right]_2^4 = \left(\frac{-1}{4} + \frac{1}{6}\right) - \left(\frac{-1}{2} + \frac{1}{4}\right) = \frac{1}{6}$

5. Compute $\int_0^2 \frac{1}{x^2+4}\,dx$.

a. $\frac{\pi}{16}$

b. $\frac{\pi}{8}$ correct choice

c. $\frac{\pi}{4}$

d. π

e. 2π

Solution: Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\,d\theta$. So:

$$\int_0^2 \frac{1}{x^2+4}\,dx = \int_0^{\pi/4} \frac{1}{4\tan^2\theta+4} 2\sec^2\theta\,d\theta = \frac{1}{2}\int_0^{\pi/4} 1\,d\theta = \left[\frac{\theta}{2}\right]_0^{\pi/4} = \frac{\pi}{8}$$

6. The integral $\int_1^\infty \frac{\sin^2 x}{x^3}\,dx$

a. converges to $\frac{1}{2}$

b. converges to 3

c. converges to $\frac{3}{4}$

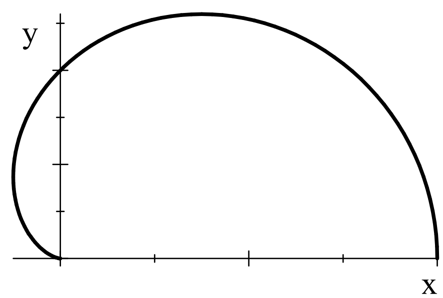
d. converges but not to $\frac{1}{2}$, 3 nor $\frac{3}{4}$ correct choice

e. diverges

Solution: $0 \le \sin^2 x \le 1$

$$\int_1^\infty \frac{\sin^2 x}{x^3}\,dx < \int_1^\infty \frac{1}{x^3}\,dx = \left[\frac{-1}{2x^2}\right]_1^\infty = 0 + \frac{1}{2} = \frac{1}{2} < \frac{3}{4} < 3$$

7. Find the area inside the upper half of the cardioid, $r = 1 + \cos\theta$.



a. $\frac{\pi}{4}$

b. $\frac{3\pi}{8}$

c. $\frac{\pi}{2}$

d. $\frac{3\pi}{4}$ correct choice

e. $\frac{3\pi}{2}$

Solution: $A = \int_0^\pi \frac{1}{2}r^2\,d\theta = \int_0^\pi \frac{1}{2}(1+\cos\theta)^2\,d\theta = \frac{1}{2}\int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\,d\theta$

$$= \frac{1}{2}\int_0^\pi \left(1 + 2\cos\theta + \frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^\pi = \frac{1}{2}\left(\pi + \frac{\pi}{2}\right) = \frac{3\pi}{4}$$

8. Find the mass of a 9 cm bar whose mass density $\delta(x) = \dfrac{1}{(x+1)^2}$ if x is measured from one end of the bar.

- **a**. $\dfrac{1}{10}$
- **b**. $\dfrac{9}{10}$ correct choice
- **c**. $\dfrac{11}{10}$
- **d**. $\dfrac{1}{100}$
- **e**. $\dfrac{99}{100}$

Solution: $M = \int_0^9 \dfrac{1}{(x+1)^2}\,dx = \left[\dfrac{-1}{x+1}\right]_0^9 = \dfrac{-1}{10} - \dfrac{-1}{1} = \dfrac{9}{10}$

9. Consider the series $S = \sum_{n=2}^{\infty} a_n$. If the k^{th} partial sum is $S_k = \sum_{n=2}^{k} a_n = \dfrac{n}{2n+1}$, then the sum is

- **a**. $S = \dfrac{1}{4}$
- **b**. $S = \dfrac{2}{5}$
- **c**. $S = \dfrac{1}{2}$ correct choice
- **d**. $S = \dfrac{3}{5}$
- **e**. $S = \dfrac{3}{4}$

Solution: $S = \lim_{n\to\infty} S_k = \lim_{n\to\infty} \dfrac{n}{2n+1} = \dfrac{1}{2}$

10. Consider the series $S = \sum_{n=2}^{\infty} a_n$. If the k^{th} partial sum is $S_k = \sum_{n=2}^{k} a_n = \dfrac{n}{2n+1}$, then the a_4 term of the series is

- **a**. $a_4 = \dfrac{1}{63}$ correct choice
- **b**. $a_4 = \dfrac{1}{99}$
- **c**. $a_4 = \dfrac{3}{7}$
- **d**. $a_4 = \dfrac{4}{9}$
- **e**. $a_4 = \dfrac{5}{11}$

Solution:
$S_4 = a_2 + a_3 + a_4$ and $S_3 = a_2 + a_3$. So $a_4 = S_4 - S_3 = \dfrac{4}{2\cdot 4+1} - \dfrac{3}{2\cdot 3+1} = \dfrac{4}{9} - \dfrac{3}{7} = \dfrac{1}{63}$

11. A ball is dropped from 64 inches. Each time it bounces, the it reaches half the height of the previous bounce. What is the total vertical distance it travels in an infinite number of bounces?

- **a**. 256
- **b**. 192 correct choice
- **c**. 128
- **d**. 96
- **e**. 64

Solution: The ball goes down 64, up 32, down 32, up 16, down 16, etc. So

$$S = 64 + 2\sum_{n=0}^{\infty} 32\left(\frac{1}{2}\right)^n = 64 + 2\frac{32}{1-\frac{1}{2}} = 64 + \frac{128}{2-1} = 192$$

12. Compute: $\displaystyle\sum_{n=1}^{\infty}\left(\sec\frac{\pi}{4n} - \sec\frac{\pi}{4(n+1)}\right)$

- **a**. $\dfrac{1}{\sqrt{2}}$
- **b**. $\dfrac{1}{\sqrt{2}} - 1$
- **c**. $\sqrt{2}$
- **d**. $\sqrt{2} - \dfrac{1}{\sqrt{2}}$
- **e**. $\sqrt{2} - 1$ correct choice

Solution: The partial sum is

$$\begin{aligned} S_k &= \sum_{n=1}^{k}\left(\sec\frac{\pi}{4n} - \sec\frac{\pi}{4(n+1)}\right) \\ &= \left(\sec\frac{\pi}{4} - \sec\frac{\pi}{8}\right) + \left(\sec\frac{\pi}{8} - \sec\frac{\pi}{12}\right) + \cdots + \left(\sec\frac{\pi}{4k} - \sec\frac{\pi}{4(k+1)}\right) \\ &= \sec\frac{\pi}{4} - \sec\frac{\pi}{4(k+1)} \end{aligned}$$

So the sum is $S = \lim\limits_{k\to\infty} S_k = \lim\limits_{k\to\infty}\left(\sec\frac{\pi}{4} - \sec\frac{\pi}{4(k+1)}\right) = \sec\frac{\pi}{4} - \sec 0 = \sqrt{2} - 1$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. $(22$ points$)$ Compute each limit:

a. $(6$ pts$)$ $\lim_{n\to\infty}\left(\frac{n^2}{n-1}-\frac{n^2}{n+1}\right)$

Solution: Put over common denominator:

$$\lim_{n\to\infty}\left(\frac{n^2}{n-1}-\frac{n^2}{n+1}\right) = \lim_{n\to\infty}\left(\frac{n^2(n+1)-n^2(n-1)}{(n-1)(n+1)}\right) = \lim_{n\to\infty}\frac{2n^2}{n^2-1} = 2$$

b. $(8$ pts$)$ $\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)$

Solution: Multiply and divide by conjugate:

$$\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right)\frac{\sqrt{n+\sqrt{n}}+\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \lim_{n\to\infty}\frac{n+\sqrt{n}-n}{\sqrt{n+\sqrt{n}}+\sqrt{n}} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+\sqrt{n}}+\sqrt{n}}$$

Divide numerator and denominator by $\sqrt{n}$:

$$\lim_{n\to\infty}\left(\sqrt{n+\sqrt{n}}-\sqrt{n}\right) = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{\sqrt{n}}}+1} = \frac{1}{2}$$

c. $(8$ pts$)$ $\lim_{n\to\infty} n^{1/n}$

Solution: Insert $e^{\ln}$:

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\ln n^{1/n}} = e^{\lim_{n\to\infty}\frac{\ln n}{n}} \overset{l'H}{=} e^{\lim_{n\to\infty}\frac{\frac{1}{n}}{1}} = e^0 = 1$$

14. (22 points) Consider the recursively defined sequence for which $a_1 = 4$ and $a_{n+1} = \sqrt{10a_n - 16}$.
Determine whether the limit of the sequence exists and if it exists, find the limit.
Follow these steps:

a. (3 pts) Find the first 3 terms:

Solution: $a_1 =$ ____4___ $a_2 =$ __$\sqrt{24}$__ $a_3 =$ ___$\sqrt{10\sqrt{24} - 16}$__

b. (4 pts) Assuming the limit exists, find the possible values of the limit.

Solution: Let $L = \lim_{n\to\infty} a_n$. Then $L = \sqrt{10L - 16}$, or $L^2 = 10L - 16$ or $L^2 - 10L + 16 = 0$ or $(L-2)(L-8) = 0$ So $L = 2$ or $L = 8$.

c. (5 pts) State whether you want to show the sequence is increasing or decreasing.
Prove it using induction.

Solution: We want to prove the sequence is increasing, i.e. $a_{n+1} > a_n$.
Initialization: $a_2 = \sqrt{24} > \sqrt{16} = 4 = a_1$.
Induction: Assume $a_{k+1} > a_k$. Then $10a_{k+1} > 10a_k$ and $10a_{k+1} - 16 > 10a_k - 16$.
and $\sqrt{10a_{k+1} - 16} > \sqrt{10a_k - 16}$ or $a_{k+2} > a_{k+1}$.
So $a_{n+1} > a_n$.

d. (5 pts) State whether you want to show the sequence is bounded above or below.
Prove it using induction.

Solution: We want to prove the sequence is bounded above by 8, i.e. $a_n < 8$.
Initialization: $a_1 = 4 < 8$ $a_2 = \sqrt{24} < \sqrt{64} = 8$.
Induction: Assume $a_k < 8$. Then $10a_k < 80$ and $10a_k - 16 < 64$
and $\sqrt{10a_k - 16} < 8$ or $a_{k+1} < 8$.
So $a_n < 8$.

e. (5 pts) Name the theorem which implies the sequence converges or diverges. State the part of the theorem you need to apply. Apply it. State the limit you get and why.

Solution: The Bounded Monotonic Sequence Theorem states that if a sequence is increasing and bounded above, then the sequence converges. So this sequence has a limit, and it must be 2 or 8. Since it increases from 4, the limit must be 8.