Name____________________________________

MATH 172 Exam 2 Spring 2020

Sections 501 Solutions P. Yasskin

Multiple Choice: (Points indicated. No part credit.)

1. (1 points) An Aggie does not lie, cheat or steal or tolerate those who do.

True [X] False [ ]

2. (1 points) Each answer is one of the following:

a rational number in lowest terms, e.g. $-\frac{217}{5}$ which is entered as "-217/5"

a rational number in lowest terms times π, e.g. $\frac{217}{5}\pi$ which is entered as "217/5pi"

positive infinity, ∞, which entered as "infinity"

negative infinity, $-\infty$, which entered as "-infinity"

convergent, which entered as "convergent"

divergent, which entered as "divergent"

Do not leave any spaces. Do not use decimals.

I read this.

True [X] False [ ]

3. (5 points) Compute $\int_0^1 \ln x\,dx$. If divergent, enter "infinity" or "-infinity".

a. $-\infty$

b. -1 correct choice

c. 0

d. 1

e. ∞

Solution: $\int_0^1 \ln x\,dx = \left[x\ln x - x\right]_0^1$.

The upper limit is $1\ln 1 - 1 = 0 - 1 = -1$.

When we evaluate $x\ln x$ at $x = 0$, we get the indeterminate form $0 \cdot (-\infty)$.

So we need to use a limit. We rewrite the limit as a fraction and use l'Hospital's rule.

$$\int_0^1 \ln x\,dx = \left[x\ln x - x\right]_{0^+}^1 = -1 - \lim_{x\to 0^+}(x\ln x - x) = -1 - \lim_{x\to 0^+}\frac{\ln x - 1}{\frac{1}{x}} \overset{l'H}{=} -1 - \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= -1 + \lim_{x\to 0^+} x = -1$$

1-10	/50	12	/16
11	/16	13	/21
		Total	/103

4. (5 points) Compute $\int_0^{\infty} \frac{e^x}{1+e^{2x}}\, dx$. If divergent, enter "infinity" or "-infinity".

a. 0

b. $\frac{1}{4}\pi$ correct choice

c. $\frac{1}{2}\pi$

d. π

e. ∞

Solution: Let $u = e^x$. Then $du = e^x\, dx$. So

$$\int_0^{\infty} \frac{e^x}{1+e^{2x}}\, dx = \int_1^{\infty} \frac{1}{1+u^2}\, du = \Big[\arctan u\Big]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

5. (5 points) Compute $\int_{-8}^{8} \frac{1}{x^{5/3}}\, dx$. If divergent, enter "divergent".

a. 0

b. $\frac{3}{8}$

c. $\frac{-3}{8}$

d. $\frac{6}{8}$

e. divergent correct choice

Solution: $\int_{-8}^{8} \frac{1}{x^{5/3}}\, dx = \int_{-8}^{0} \frac{1}{x^{5/3}}\, dx + \int_{0}^{8} \frac{1}{x^{5/3}}\, dx$

$$\int_{-8}^{0} \frac{1}{x^{5/3}}\, dx = \left[\frac{-3}{2x^{2/3}}\right]_{-8}^{0^-} = \left(\frac{-3}{2(0^-)^{2/3}}\right) - \left(\frac{-3}{2(-8)^{2/3}}\right) - \infty + \frac{3}{8} = -\infty$$

$$\int_{0}^{8} \frac{1}{x^{5/3}}\, dx = \left[\frac{-3}{2x^{2/3}}\right]_{0^+}^{8} = \left(\frac{-3}{2(8)^{2/3}}\right) - \left(\frac{-3}{2(0^+)^{2/3}}\right) = -\frac{3}{8} + \infty = \infty$$

Since at least one of these is divergent, the total is divergent.

6. (5 points) What is the **total** number of coefficients in the general partial fraction expansion of

$$\frac{x^5+x^4}{(x-2)(x-3)^4(x^2+4)^3}$$

For example $\frac{Bx+C}{(x^2+9)^4}$ has 2 coefficients.

a. 16

b. 11 correct choice

c. 8

d. 7

e. 4

Solution: $\frac{x^5+x^4}{(x-2)(x-3)^4(x^2+4)^3}$

$$= \frac{A}{x-2} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{D}{(x-3)^3} + \frac{E}{(x-3)^4} + \frac{Fx+G}{(x^2+4)} + \frac{Hx+I}{(x^2+4)^2} + \frac{Jx+K}{(x^2+4)^3}$$

7. $\left(5 \text{ points}\right)$ The base of a solid is the region between $y = x^2$ and $y = 4$. The crosssectons perpendicular to the y axis are squares. Find its volume.

a. 2

b. 4

c. 8

d. 16

e. 32 correct choice

Solution: This is a y integral. The limits are $y = 0$ and $y = 4$. The width of a horizontal slice is $w = 2\sqrt{y}$. So the volume is

$$V = \int_0^4 w^2\,dy = \int_0^4 \left(2\sqrt{y}\right)^2 dy = \int_0^4 4y\,dy = \left[2y^2\right]_0^4 = 32$$

8. $\left(5 \text{ points}\right)$ The region between $y = x^2$ and $y = 4$ is rotated about the x axis. Find the volume.

a. 8π

b. 16π

c. $\frac{64}{15}\pi$

d. $\frac{128}{15}\pi$

e. $\frac{256}{15}\pi$ correct choice

Solution: This is an x integral. Slices are vertical and rotate into washers. The big radius is $R = 4$. The small radius is $r = x^2$. The limits are $x = -2$ to $x = 2$. So the volume is:

$$V = \int_{-2}^{2} \pi R^2 - \pi r^2\,dx = \pi \int_{-2}^{2} 16 - x^4\,dx = \pi\left[16x - \frac{x^5}{5}\right]_{-2}^{2}$$
$$= 2\pi\left(32 - \frac{32}{5}\right) = 64\pi\left(1 - \frac{1}{5}\right) = \frac{256}{15}\pi$$

9. $\left(5 \text{ points}\right)$ The region between $y = x^2$ and $y = 4$ is rotated about the y axis. Find the volume.

a. 8π correct choice

b. 16π

c. $\frac{64}{15}\pi$

d. $\frac{128}{15}\pi$

e. $\frac{256}{15}\pi$

Solution: This is an x integral. Slices are vertical and rotate into cylinders. The radius is $r = x$. The height is $h = 4 - x^2$. The limits are $x = 0$ to $x = 2$ because otherwise we would double count the volume. So the volume is:

$$V = \int_0^2 2\pi r h\,dx = 2\pi \int_0^2 x(4 - x^2)\,dx = 2\pi\left[2x^2 - \frac{x^4}{4}\right]_0^2 = 2\pi[8 - 4] = 8\pi$$

10. (5 points) Duke Skywater just arrived on the planet Corona. He measured that it takes 48 J of work to lift a 3 kg weight by 4 m. What is the acceleration of gravity on the surface of Corona? (Do not enter units.)

a. $3\ \dfrac{\text{m}}{\text{sec}^2}$

b. $4\ \dfrac{\text{m}}{\text{sec}^2}$ correct choice

c. $12\ \dfrac{\text{m}}{\text{sec}^2}$

d. $60\ \dfrac{\text{m}}{\text{sec}^2}$

e. $64\ \dfrac{\text{m}}{\text{sec}^2}$

Solution: $W = mgh \qquad 48 = 3g4 \qquad g = 4$

11. (5 points) A 200 foot chain weighs $\delta = 2\ \dfrac{\text{lb}}{\text{foot}}$. It is hanging from the top of a 200 foot tall building. How much work is done to pull it up to the top of the building.

a. 5000

b. 10000

c. 20000

d. 40000 correct choice

e. 80000

Solution: Put the 0 of the y-axis at the top of the building and measure y downward. The piece of rope of length dy feet at a distance of y feet from the top is lifted a distance $D = y$ feet. Its weight is $dF = \delta\,dy = 2\,dy$. So the work done to lift the rope is

$$W = \int_0^{200} D\,dF = \int_0^{200} y\,2\,dy = \left[y^2\right]_0^{200} = 40000$$

12. (5 points) A weight is attached to a spring whose rest position is at $x_o = 3$ m. It takes 24 J of work to move the weight from $x = 3$ m to $x = 7$ m. How much work (in Joules) is needed to stretch the weight from $x = 6$ m to $x = 9$ m? (The answer is positive. Do not write the units.)

a. 9 J

b. $\dfrac{27}{2}$ J

c. 27 J

d. $\dfrac{81}{2}$ J correct choice

e. 81 J

Solution: $F = k(x - x_o)$

$$W = \int F\,dx = \int_3^7 k(x-3)\,dx = \left[k\frac{(x-3)^2}{2}\right]_3^7 = 8k = 24 \qquad k = 3$$

$$W = \int F\,dx = \int_6^9 3(x-3)\,dx = \left[\frac{3}{2}(x-3)^2\right]_6^9 = \frac{3}{2}(36-9) = \frac{81}{2}$$

13. (21 points) An oil tank is a cylinder 3 m in radius and 6 m long. Its axis is horizontal. It is filled to a depth of 4 m above the **bottom** of the tank. The oil is flowing out a spout which is 1 m above the **bottom** of the tank. How much work is done by gravity to lower the depth to 2 m above the **bottom** of the tank? Take the density of oil and to be δ and the acceleration of gravity to be g (no numbers for δ and g).

a. Where should you put the 0 of the y-axis? Take y to be positive upward.

- **i**. at the spout
- **ii**. at the top of the tank
- **iii**. at the center of the tank   correct choice
- **iv**. at the bottom of the tank

Set up the integral for the work. It will have the form:

$$W = \boxed{\mathbf{b}}\,\delta g \int_{\boxed{\mathbf{c}}}^{\boxed{\mathbf{d}}} \left(y + \boxed{\mathbf{e}}\right)\left(\boxed{\mathbf{f}} - y^2\right)^{\boxed{\mathbf{g}}} dy$$

Identify each of the quantities in boxes:

b. coefficient: $b = 12$

c. lower limit: $c = -1$

d. upper limit: $d = 1$

e. coefficient: $e = 2$

f. coefficient: $f = 9$

g. exponent: $g = 1/2$

Solution: Assuming the 0 of the y-axis is in the center of the tank, the water at height y with thickness dy is a rectangular slab with length $L = 6$, width $W = 2x$ and height $H = dy$, where $x^2 + y^2 = 3^2$. So $x = \sqrt{9 - y^2}$. So its volume is

$$dV = 6 \cdot 2\sqrt{9 - y^2}\, dy$$

and its weight is:

$$dF == 12\delta g\sqrt{9 - y^2}\, dy$$

The spout is at height $y = -3 + 1 = -2$. So the slab of water falls a distance

$$D = y - -2 = y + 2$$

The bottom of the tank is at $y = -3$. the tank is filled to a depth of 4 m which is at $y = 1$. The oil is lowered to a depth of 2 m which is at $y = -1$. So the work is

$$W = \int D\, dF = 12\delta g \int_{-1}^{1} (y + 2)\sqrt{9 - y^2}\, dy$$

Comparing to the template:

$b = 12$ $c = -1$ $d = 1$ $e = 2$ $f = 9$ $g = 1/2$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. $(16$ points$)$ Find the coefficients in the partial fraction expansion:

$$\frac{x^3+24x^2-4x}{(x-2)(x+2)(x^2+4)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4}$$

Solution: Clear the denominator:

$x^3+24x^2-4x = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2)$

$x = 2$: $\quad 8+96-8 = A(4)(8) \qquad 96 = 32A \qquad A = 3$

$x = -2$: $\quad -8+96+8 = B(-4)(8) \qquad 96 = -32B \qquad B = -3$

$x = 0$: $\quad 0 = A(2)(4) + B(-2)(4) + D(-2)(2) = 24+24-4D \qquad D = 12$

Coeff of x^3: $\quad 1 = A+B+C = C \qquad C = 1$

$$\frac{x^3+24x^2-4x}{(x-2)(x+2)(x^2+4)} = \frac{3}{x-2} + \frac{-3}{x+2} + \frac{x+12}{x^2+4}$$

15. $(16$ points$)$ Given the partial fraction expansion

$$\frac{36x+54}{(x^2+9)(x+3)^2} = \frac{1}{x+3} + \frac{-3}{(x+3)^2} + \frac{-x+6}{x^2+9}$$

Compute $\displaystyle\int \frac{36x+54}{(x^2+9)(x+3)^2}\,dx$.

Solution:

$$\int \frac{1}{x+3}\,dx = \ln|x+3| + C_1$$

$$\int \frac{-3}{(x+3)^2}\,dx = \frac{3}{x+3} + C_2$$

$$\int \frac{-x}{x^2+9}\,dx = -\frac{1}{2}\ln|x^2+9| + C_3$$

In the last integral, let $x = 3\tan\theta \qquad dx = 3\sec^2\theta\,d\theta$.

$$\int \frac{6}{x^2+9}\,dx = \int \frac{6}{9\tan^2\theta+9}3\sec^2\theta\,d\theta = 2\int \frac{\sec^2\theta}{\tan^2\theta+1}\,d\theta$$
$$= 2\int 1\,d\theta = 2\theta = 2\arctan\frac{x}{3} + C_4$$

So

$$\int \frac{36x+54}{(x^2+9)(x+3)^2}\,dx = \ln|x+3| + \frac{3}{x+3} - \frac{1}{2}\ln|x^2+9| + 2\arctan\frac{x}{3}x + C$$