Name________________________________

MATH 172H Exam 3 Spring 2020

Sections 200 Solutions P. Yasskin

Multiple Choice: (Points indicated. No part credit.)

1. (1 points) An Aggie does not lie, cheat or steal or tolerate those who do.

True $\boxed{X}$ False $\boxed{}$

2. (1 points) Each answer is one of the following or a sum of these:

a rational number in lowest terms, e.g. $-\dfrac{217}{5}$ which is entered as "-217/5"

a rational number in lowest terms times π, e.g. $\dfrac{217}{5}\pi$ which is entered as "217/5pi"

exponentials such as e^4 or $3^{12/5}$ which are entered as "e^4" or "3^(12/5)"

positive infinity, ∞, which is entered as "infinity"

negative infinity, $-\infty$, which is entered as "-infinity"

convergent, which is entered as "convergent"

divergent, which is entered as "divergent"

Do not leave any spaces. Do not use decimals.

I read this.

True $\boxed{X}$ False $\boxed{}$

3. (4 points) Compute $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 4}{2^n}$.

a. 0

b. $-\dfrac{4}{3}$ correct choice

c. 4

d. -4

e. ∞

Solution: Geometric $a = \dfrac{-4}{2} = -2 \qquad r = -\dfrac{1}{2} \qquad S = \dfrac{a}{1-r} = \dfrac{-2}{1+\frac{1}{2}} = \dfrac{-4}{3}$

1-9	/30	13	/18
10-11	/28	14	/8
12	/18	Total	/102

4. (4 points) Compute $\sum_{n=1}^{\infty}\left(\frac{n}{3n-1} - \frac{n+1}{3n+2}\right)$.

- **a**. 0
- **b**. $\frac{1}{3}$
- **c**. $\frac{2}{3}$
- **d**. $\frac{1}{6}$ correct choice
- **e**. ∞

Solution: Telescoping

$$S_k = \sum_{n=1}^{\infty}\left(\frac{n}{3n-1} - \frac{n+1}{3n+2}\right) = \left(\frac{1}{2} - \frac{2}{5}\right) + \left(\frac{2}{5} - \frac{3}{8}\right) + \cdots + \left(\frac{k}{3k-1} - \frac{k+1}{3k+2}\right) = \frac{1}{2} - \frac{k+1}{3k+2}$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{k+1}{3k+2}\right) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

5. (4 points) Compute $\lim_{n\to\infty}\left(\sqrt{n^6+5n^2} - \sqrt{n^6-4n^3}\right)$.

- **a**. $-\infty$
- **b**. -4
- **c**. 2 correct choice
- **d**. 9
- **e**. ∞

Solution: $\lim_{n\to\infty}\left(\sqrt{n^6+5n^2} - \sqrt{n^6-4n^3}\right)\frac{\sqrt{n^6+5n^2} + \sqrt{n^6-4n^3}}{\sqrt{n^6+5n^2} + \sqrt{n^6-4n^3}} = \lim_{n\to\infty}\frac{(n^6+5n^2)-(n^6-4n^3)}{\sqrt{n^6+5n^2} + \sqrt{n^6-4n^3}}$

$$= \lim_{n\to\infty}\frac{5n^2+4n^3}{\sqrt{n^6+5n^2} + \sqrt{n^6-4n^3}} \cdot \frac{\frac{1}{n^3}}{\frac{1}{n3}} = \lim_{n\to\infty}\frac{\frac{5}{n}+4}{\sqrt{1+5n^{-4}} + \sqrt{1-2n^{-3}}} = \frac{4}{2} = 2$$

6. (4 points) Compute $\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{\frac{4}{\ln n}}$. If divergent, enter "infinity" or "-infinity".

- **a**. e^{-8} correct choice
- **b**. e^{-2}
- **c**. e^{2}
- **d**. e^{8}
- **e**. ∞

Solution: $\lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{\frac{4}{\ln n}} = e^{\ln \lim_{n\to\infty}\left(\frac{1}{n^2}\right)^{\frac{4}{\ln n}}} = \exp \lim_{n\to\infty} \ln\left(\frac{1}{n^2}\right)^{\frac{4}{\ln n}}$

$$= \exp \lim_{n\to\infty}\frac{4}{\ln n}\ln(n^{-2}) = \exp \lim_{n\to\infty}\frac{-8}{\ln n}\ln(n) = e^{-8}$$

7. (4 points) Compute $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n}$. If divergent, enter "infinity" or "-infinity".

a. $\sin 2$

b. $\sin 2 - 1$

c. $\cos 2$

d. $\cos 2 - 1$ correct choice

e. ∞

Solution: A standard Maclaurin series is $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$.

At $x = 2$ this says $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} = \cos 2$. Our series starts at $n = 1$. So

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} 2^{2n} - 1 = \cos 2 - 1$$

8. (4 points) If $S = \sum_{n=1}^{\infty} a_n$ and $S_k = \frac{k}{2k+1}$, then

a. $a_4 = \frac{4}{9}$

b. $a_4 = \frac{3}{7}$

c. $a_4 = \frac{1}{2}$

d. $a_4 = \frac{1}{63}$ correct choice

e. $a_4 = \frac{4}{21}$

Solution: $S_4 = a_1 + a_2 + a_3 + a_4 = \frac{4}{9}$ $S_3 = a_1 + a_2 + a_3 = \frac{3}{7}$ $a_4 = S_4 - S_3 = \frac{4}{9} - \frac{3}{7} = \frac{1}{63}$

9. (4 points) If the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ is approximated by the 999^{th} partial sum

$S_{999} = \sum_{n=1}^{999} \frac{(-1)^{n+1}}{n^2} \approx 0.901542677870445 71280$, how many digits of accuracy are guaranteed in this approximation? For example, if the error is $|E_{999}| < 10^{-5}$, then only the digits 0.9015 are accurate, and you would answer 4.

a. 4,

b. 5 correct choice

c. 10

d. 1000

e. 1000000

Solution: Since this is an alternating, decreasing series, the error is less than the absolute value of the next term which is $|E_{999}| < \frac{1}{1000^2} = 10^{-6}$. So the approximation is good to 5 terms.

10. $\left(14\right.$ points$\left.\right)$ The series $\sum_{n=2}^{\infty} \frac{1}{n-1}$ can be shown to diverge by which of the following Convergence Tests? Check Yes for all that work; check No for all that don't work.

a. n^{th}-Term test for Divergence:

[ ] Yes [X] No $\quad \lim_{n\to\infty} \frac{1}{n-1} = 0$ Test Fails

b. Integral Test:

[X] Yes [ ] No $\quad \int_2^{\infty} \frac{1}{n-1}\,dn = \left[\ln(n-1)\right]_2^{\infty} = \infty$

c. p-Series Test:

[X] Yes [ ] No $\quad \sum_{n=2}^{\infty} \frac{1}{n-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$ p-series with $p = 1$ harmonic

d. Simple Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n}$:

[X] Yes [ ] No $\quad \frac{1}{n-1} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges

e. Limit Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n}$:

[X] Yes [ ] No $\quad \lim_{n\to\infty} \frac{n}{n-1} = 1$ and $0 < 1 < \infty$

f. Ratio Test:

[ ] Yes [X] No $\quad \lim_{n\to\infty} \frac{n-1}{(n+1)-1} = 1$ Test Fails

g. Alternating Series Test:

[ ] Yes [X] No This series is not alternating.

11. $\left(14 \text{ points}\right)$ The series $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$ can be shown to converge by which of the following Convergence Tests? Check Yes for all that work; check No for all that don't work.

a. n^{th}-Term test for Divergence:

[_____] Yes [__X__] No $\lim_{n \to \infty} \frac{1}{n^2 - 1} = 0$ Test Fails

b. Integral Test:

[__X__] Yes [_____] No $\int_2^{\infty} \frac{1}{n^2 - 1} dn = \left[\frac{1}{2} \ln\left(\frac{n-1}{n+1} \right) \right]_2^{\infty} = \frac{1}{2} \ln 3 < \infty$

c. p-Series Test:

[_____] Yes [__X__] No This is not a p-series.

d. Simple Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n^2}$:

[_____] Yes [__X__] No $\frac{1}{n^2 - 1} > \frac{1}{n^2}$ Wrong inequality.

e. Limit Comparison Test comparing to $\sum_{n=2}^{\infty} \frac{1}{n^2}$:

[__X__] Yes [_____] No $\lim_{n \to \infty} \frac{n^2}{n^2 - 1} = 1$ and $0 < 1 < \infty$

f. Ratio Test:

[_____] Yes [__X__] No $\lim_{n \to \infty} \frac{n^2 - 1}{(n+1)^2 - 1} = 1$ Test Fails

g. Alternating Series Test:

[_____] Yes [__X__] No This series is not alternating.

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (18 points) Find the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-2)^n}{1+\sqrt{n}}(x-3)^n$.

a. Find the radius of convergence and state the open interval of absolute convergence.

$R =$ ____. Absolutely convergent on (_____,_____).

Solution: To find the radius, we use the Ratio Test. $|a_n| = \frac{2^n|x-3|^n}{1+\sqrt{n}}$ $|a_{n+1}| = \frac{2^{n+1}|x-3|^{n+1}}{1+\sqrt{n+1}}$

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^{n+1}|x-3|^{n+1}}{1+\sqrt{n+1}} \frac{1+\sqrt{n}}{2^n|x-3|^n} = 2|x-3| \lim_{n\to\infty} \frac{1+\sqrt{n}}{1+\sqrt{n+1}} = 2|x-3| < 1$$

$|x-3| < \frac{1}{2}$ So $R = \frac{1}{2}$. Absolutely convergent on $\left(\frac{5}{2}, \frac{7}{2}\right)$

b. Check the **Left** Endpoint:

$x =$_____ The series becomes_________________________ Circle one:

Reasons: Convergent

Solution: $x = \frac{5}{2}$: $\sum_{n=1}^{\infty} \frac{(-2)^n}{1+\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ $\boxed{\text{Divergent}}$

Compare this to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a p-series with $p = \frac{1}{2} < 1$ and so diverges.

We can't use the Simple Comparison Test because $\frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}}$. So we compute:

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} \cdot \frac{\sqrt{n}}{1} = \lim_{n\to\infty} \frac{1}{\frac{1}{\sqrt{n}}+1} = 1.$$

Since $0 < L = 1 < \infty$, the series $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ diverges by the Limit Comparison Test.

c. Check the **Right** Endpoint:

$x =$_____ The series becomes_________________________ Circle one:

Reasons: $\boxed{\text{Convergent}}$

Solution: $x = \frac{7}{2}$: $\sum_{n=1}^{\infty} \frac{(-2)^n}{1+\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{1+\sqrt{n}}$ Divergent

This converges by the Alternating Series Test because $\frac{1}{1+\sqrt{n}}$ is positive, decreasing and $\lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0$.

d. State the Interval of Convergence. Interval= ____________

Solution: The Interval of Convergence.is: $\left(\frac{5}{2}, \frac{7}{2}\right]$

13. $(18$ points$)$ Determine whether the recursively defined sequence $a_1 = \sqrt{6}$ and $a_{n+1} = \dfrac{(a_n)^2 + 24}{10}$ is convergent or divergent. If convergent, find the limit. If divergent, say infinity or -infinity.

a. Find the first 3 terms: $a_1 =$ ______ $a_2 =$ ______ $a_3 =$ ______

Solution: $a_1 =$ ___$\sqrt{6}$___ $a_2 =$ ___3___ $a_3 =$ ___3.3___

b. Assuming the limit $\lim\limits_{n\to\infty} a_n$ exists, find the possible limits.

Solution: Assume $\lim\limits_{n\to\infty} a_n = L$. Then $\lim\limits_{n\to\infty} a_{n+1} = L$ also. From the recursion relation:

$$L = \frac{L^2 + 24}{10} \qquad L^2 - 10L + 24 = 0 \qquad (L-4)(L-6) = 0 \qquad L = 4, 6$$

c. Prove the sequence is increasing or decreasing (as appropriate).

Solution: From the first 3 terms, we expect the sequence is increasing. So we want to prove $a_{n+1} > a_n > 0$.
Initialization Step: Since $9 > 6 > 0$, we know $a_2 = 3 = \sqrt{9} > \sqrt{6} = a_1 > 0$
Induction Step: Assume $a_{k+1} > a_k > 0$. We need to prove $a_{k+2} > a_{k+1} > 0$.
Proof: We need > 0 so we can square both sides of an inequality.

$$a_{k+1} > a_k > 0 \quad\Rightarrow\quad (a_{k+1})^2 > (a_k)^2 > 0 \quad\Rightarrow\quad \frac{(a_{k+1})^2 + 24}{10} > \frac{(a_k)^2 + 24}{10} > \frac{24}{10} > 0$$

$$\Rightarrow\quad a_{k+2} > a_{k+1} > 0$$

d. Prove the sequence is bounded or unbounded above or below (as appropriate).

Solution: The sequence starts at $\sqrt{6} < 4$, and increases and has a limit of 4 or 6 if it exists. So we try to prove $a_n < 4$.
Initialization Step: $a_1 = \sqrt{6} < 4$
Induction Step: Assume $a_k < 4$. We need to prove $a_{k+1} < 4$.
Proof:

$$a_k < 4 \quad\Rightarrow\quad (a_k)^2 < 16 \quad\Rightarrow\quad (a_k)^2 + 24 < 40 \quad\Rightarrow\quad \frac{(a_k)^2 + 24}{10} < 4 \quad\Rightarrow\quad a_{k+1} < 4$$

e. State whether the sequence is convergent or divergent and name the theorem. If convergent, determine the limit. If divergent, determine if it is infinity or -infinity.

Solution: The sequence is convergent by the Bounded Monotonic Sequence Theorem. Since the limit must be 4 or 6 and it increases from $\sqrt{6}$ and is bounded above by 4, the limit must be $\lim\limits_{n\to\infty} a_n = 4$.

14. (8 points) A ball is dropped from a height of 72 feet. Each time it bounces it reaches a height which is $\frac{1}{3}$ of the height on the previous bounce. What is the total distance travelled by the ball (with an infinite number of bounces)?

Solution: The ball drops 72 ft, rises and falls 24 ft, rises and falls 8 ft, etc. The total distance is:

$$D = 72 + 2\left(24 + 8 + \frac{8}{3} + \cdots\right) = 72 + 2\sum_{n=0}^{\infty} 24\left(\frac{1}{3}\right)^n = 72 + 2\left(\frac{24}{1 - \frac{1}{3}}\right) = 72 + 2(36) = 144$$