Name________________________________

MATH 172H Final Exam Spring 2020

Sections 200 Solutions P. Yasskin

Multiple Choice: (Points indicated. No part credit.)

**1**. $\left(1 \text{ points}\right)$ An Aggie does not lie, cheat or steal or tolerate those who do.

True [X] False [ ]

**2**. $\left(1 \text{ points}\right)$ Each answer is one of the following or a sum of these:

a rational number in lowest terms, e.g. $-\frac{217}{5}$ which is entered as "-217/5"

a rational number in lowest terms times $\pi$, e.g. $\frac{217}{5}\pi$ which is entered as "217/5pi"

square roots are entered using sqrt, e.g. $\frac{1}{2}\sqrt{3}\,\pi$ which is entered as "1/2sqrt(3)pi"

positive infinity, $\infty$, which is entered as "infinity"

negative infinity, $-\infty$, which is entered as "-infinity"

convergent, which is entered as "convergent"

divergent, which is entered as "divergent"

Do not leave any spaces. Do not use decimals.

I read this.

True [X] False [ ]

**3**. $\left(5 \text{ points}\right)$ Find the average value of $f(x) = \sin x$ on the interval $0 \le x \le \pi$.

**a**. $2$

**b**. $\pi$

**c**. $\frac{1}{\pi}$

**d**. $\frac{2}{\pi}$ correct choice

**e**. $\frac{1}{2\pi}$

**Solution**: $f_{\text{ave}} = \frac{1}{\pi}\int_0^\pi \sin x\,dx = \frac{1}{\pi}[-\cos x]_0^\pi = \frac{1}{\pi}(--1--1) = \frac{2}{\pi}$

| | | | |
|---|---|---|---|
| 1-9 | /37 | 12 | /15 |
| 10 | /25 | 13 | /10 |
| 11 | /15 | Total | /102 |

**4**. (5 points) The base of a solid is a circle of radius $3$. The cross sections perpendicular to the $x$-axis are squares. Find the volume of the solid.

- **a**. $18\pi$
- **b**. $36\pi$
- **c**. $36$
- **d**. $72$
- **e**. $144$ correct choice

This is a $x$-integral. The side of the square is $s = 2\sqrt{9-x^2}$. So the area is $A = s^2 = 4(9-x^2)$.

So the volume is $V = \int_{-3}^{3} A\,dx = \int_{-3}^{3} 4(9-x^2)\,dx = 4\left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 8(27-9) = 144$

**5**. (5 points) The series $\sum_{n=2}^{\infty} \frac{(-1)^2}{n^{1/2}+n^{1/3}}$ is

- **a**. absolutely convergent
- **b**. conditionally convergent correct choice
- **c**. divergent
- **d**. absolutely divergent
- **e**. conditionally divergent

**Solution**: The original series is an alternating, decreasing series and $\lim_{n\to\infty} \frac{1}{\sqrt{n}+n^2} = 0.$

So the series converges. The related absolute series is $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}+n^{1/3}}$ which converges by limit comparison with $\sum_{n=2}^{\infty} \frac{1}{n^{1/2}}$. So the series is conditionally convergent.

**6**. (5 points) Compute $\lim_{x\to 0} \frac{\sin(2x^3)-2x^3}{x^9}$. If divergent, enter "infinity" or "-infinity".

- **a**. $-\infty$
- **b**. $-\frac{4}{3}$ correct choice
- **c**. $\frac{4}{3}$
- **d**. $\frac{8}{3}$
- **e**. $\infty$

**Solution**: $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \qquad \sin 2x^3 = 2x^3 - \frac{2^3x^9}{6} + \frac{2^5x^{15}}{120} - \cdots$

$$\lim_{x\to 0} \frac{\sin(2x^3)-2x^3}{x^9} = \lim_{n\to\infty} \frac{\left(2x^3 - \frac{2^3x^9}{6} + \frac{2^5x^{15}}{120} - \cdots\right) - 2x^3}{x^9} = \lim_{n\to\infty} \frac{-\frac{2^3x^9}{6} + \frac{2^5x^{15}}{120} - \cdots}{x^9}$$

$$= \lim_{n\to\infty}\left(-\frac{2^3}{6} + \frac{2^5x^6}{120} - \cdots\right) = -\frac{4}{3}$$

7. (5 points) Compute $\left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\cdots+\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right)+\cdots$
If divergent, enter "infinity" or "-infinity".

a. $-\frac{1}{2}$ correct choice

b. $\frac{1}{2}$

c. $1$

d. $2$

e. $\infty$

**Solution**: $S_k=\sum_{n=1}^{k}\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right)=\left(\frac{1}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{4}\right)+\cdots+\left(\frac{k}{k+1}-\frac{k+1}{k+2}\right)=\frac{1}{2}-\frac{k+1}{k+2}$

$S=\lim_{k\to\infty}\left(\frac{1}{2}-\frac{k+1}{k+2}\right)=-\frac{1}{2}$

8. (5 points) Compute $\int_0^{\pi/4}\tan^4\theta\sec^2\theta\,d\theta$

a. $125$

b. $625$

c. $\frac{4}{5}$

d. $\frac{1}{25}$

e. $\frac{1}{5}$ correct choice

**Solution**: Let $u=\tan\theta \qquad du=\sec^2\theta\,d\theta$

$\int_0^{\pi/4}\tan^4\theta\sec^2\theta\,d\theta=\int u^4\,du=\left[\frac{u^5}{5}\right]=\left[\frac{\tan^5\theta}{5}\right]_0^{\pi/4}=\frac{1}{5}$

9. (5 points) Estimate $\int_0^{0.1}3\sin(x^2)\,dx$ to within an error of $|E|<0.0001$
HINT: Use a Maclaurin series. This answer is a decimal number.

a. $0.3-\frac{(0.1)^3}{2}=0.2995$

b. $0.3$

c. $3(0.1)^2-\frac{(0.1)^6}{2}=0.0299995$

d. $(0.1)^3=.001$ correct choice

e. $(0.1)^2=.01$

**Solution**: $\sin x=x-\frac{x^3}{6}+\cdots \qquad 3\sin x^2=3x^2-\frac{x^6}{2}+\cdots \qquad \int_0^x 3\sin(x^2)\,dx=x^3-\frac{x^7}{14}+\cdots$

Using 1 term: $\int_0^{0.1}3\sin(x^2)\,dx\approx(0.1)^3=.001.$

The error is less than the next term $|E|<\frac{(0.1)^7}{14}<10^{-7}$

Work Out Problems: (Show all work on paper and upload in Part 2. Enter answers here.)

**10**. (25 points) A paper soda cup is $8$ cm tall, has a circular base of radius $4$ cm and a circular top of radius $5$ cm. So the sides are given by rotating the line $x = 4 + \frac{y}{8}$ about the $y$-axis.

**a**. Find the volume of the cup.

**Solution**: This is a $y$ integral. Each horizontal slice rotates into a disk of radius $r = x = 4 + \frac{y}{8}$. So the volume is

$$V = \int_0^8 \pi r^2\,dy = \int_0^8 \pi\left(4 + \frac{y}{8}\right)^2 dy = \frac{8\pi}{3}\left(4 + \frac{y}{8}\right)^3\Big|_0^8 = \frac{8\pi}{3}(5^3 - 4^3) = \frac{488}{3}\pi$$

**b**. Find the surface area of the sides of the cup.

**Solution**: The differential of arclength is

$$ds = \sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \sqrt{\left(\frac{1}{8}\right)^2 + 1}\,dy = \frac{\sqrt{65}}{8}\,dy$$

The radius of rotation is $r = x = 4 + \frac{y}{8}$. So the surface area is

$$A = \int_0^8 2\pi r\,ds = 2\pi\int_0^8\left(4 + \frac{y}{8}\right)\frac{\sqrt{65}}{8}\,dy = \frac{\sqrt{65}\,\pi}{4}\left[4y + \frac{y^2}{16}\right]_0^8 = 9\sqrt{65}\,\pi$$

**c**. There is soda in the cup up to $7$ cm. The density of the soda is $\delta$ and the acceleration of gravity is $g$. A $10$ cm straw is put into the cup, so that the bottom of the straw is at the bottom of the cup. Set up the integral for the work done to suck the soda through the top of the straw. Be sure to explain how you got it on your paper. Do not compute the integral. The integral will have the form:

$$W = \boxed{c}\,\delta g\int_{\boxed{d}}^{\boxed{e}}\left(\boxed{f} - y\right)\left(\boxed{g} + \boxed{h}\,y\right)^{\boxed{i}}dy$$

Identify each of the quantities in boxes:

- $c =$
- $d =$
- $e =$
- $f =$
- $g =$
- $h =$
- $i =$

**Solution**: The soda at height $y$ with thickness $dy$ is a disk of radius $r = x = 4 + \frac{y}{8}$ and height $dy$. So its volume is: $dV = \pi r^2\,dy = \pi\left(4 + \frac{y}{8}\right)^2 dy$

and its weight is: $dF == \delta g dV = \delta g\pi\left(4 + \frac{y}{8}\right)^2 dy$

This disk of soda is lifted a distance: $D = 10 - y$

There is soda between: $y = 0$ and $y = 7$.

So the work is: $W = \int D\,dF = \delta g\pi\int_0^7 (10 - y)\left(4 + \frac{y}{8}\right)^2 dy$

Comparing to the template: $c = \pi$ $d = 0$ $e = 7$ $f = 10$ $g = 4$ $h = 1/8$ $i = 2$

**11**. (15 points) Find the coefficients in the partial fraction expansion

$$\frac{6x+8}{(x-2)^2(x^2+1)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+1}$$

**Solution**: Clear the denominator:

$$6x+8 = A(x-2)(x^2+1) + B(x^2+1) + (Cx+D)(x-2)^2 \qquad (1)$$

Plug in $x = 2$: $20 = B(5)$

$$B = 4 \qquad (2)$$

Plug in $x = 0$: $8 = A(-2)(1) + B(1) + D(-2)^2 = -2A + 4 + 4D \qquad 2A - 4D = -4$

$$A - 2D = -2 \qquad (3)$$

Plug in $x = 1$: $14 = A(-1)(2) + B(2) + (C+D)(-1)^2 = -2A + 8 + C + D$

$$2A - C - D = -6 \qquad (4)$$

Coeff of $x^3$: $0 = A + C$

$$C = -A \qquad (5)$$

Plug (5) into (4):

$$3A - D = -6 \quad \text{or} \quad D = 3A + 6 \qquad (6)$$

Plug (6) into (3):

$$A - 2(3A+6) = -2 \quad \text{or} \quad -5A - 12 = -2 \quad \text{or} \quad A = -2 \qquad (7)$$

Substitute back:

$$C = 2 \qquad D = 0$$

So

$$\frac{6x+8}{(x-2)^2(x^2+1)} = \frac{-2}{x-2} + \frac{4}{(x-2)^2} + \frac{2x}{x^2+1}$$

As an alternate to (4), (6) and (7), differentiate (1):

$$6 = A(x^2+1) + A(x-2)2x + B2x + C(x-2)^2 + (Cx+D)2(x-2) \qquad (8)$$

and plug in $x = 2$: $6 = A(5) + B(4) = 5A + 16 \quad \Rightarrow \quad 5A = -10$

$$A = -2 \qquad (9)$$

**12**. (15 points) Find the 4th degree Maclaurin polynomial (Taylor polynomial at 0) for $f(x) = \sec x$. Show your derivation on paper. (Yes, you need 4 derivatives.) Enter your coefficients here.

$$\sec x = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4$$

**Solution**:

| | |
|---|---|
| $f(x) = \sec x$ | $f(0) = 1$ |
| $f'(x) = \sec x \tan x$ | $f'(0) = 0$ |
| $f''(x) = \sec x \tan^2 x + \sec^3 x$ | $f''(0) = 1$ |
| $f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$ | $f^{(3)}(0) = 0$ |
| $f^{(4)}(x) = \sec x \tan^4 x + 18\sec^3 x \tan^2 x + 5\sec^5 x$ | $f^{(4)}(0) = 5$ |

$$f(x) = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \frac{1}{4!}f^{(4)}(0)x^4$$

$$\sec x = 1 + \frac{1}{2!}x^2 + \frac{5}{4!}x^4$$

$c_0 =$ __1__ $c_1 =$ __0__ $c_2 =$ __$\frac{1}{2}$__ $c_3 =$ __0__ $c_4 =$ __$\frac{5}{24}$__

**13**. (10 points) Compute $\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{3x}\,dx$.

$x = 3\sec\theta \qquad dx = 3\sec\theta\tan\theta\,d\theta$

$$\int_3^{3\sqrt{2}} \frac{\sqrt{x^2-9}}{3x}\,dx = \int_0^{\pi/4} \frac{\sqrt{9\sec^2\theta - 9}}{9\sec\theta} 3\sec\theta\tan\theta\,d\theta = \int_0^{\pi/4} \tan^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \sec^2\theta - 1\,d\theta = \Big[\tan\theta - \theta\Big]_0^{\pi/4} = 1 - \frac{\pi}{4}$$