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15	/15	17	/20
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MATH 172H — Exam 1 — Spring 2021

Sections 200 — Solutions — P. Yasskin

Multiple Choice: (Points indicated.)

1. (5 pts) Compute $\int_0^{\pi/4} x\cos(4x)\,dx$.

a. $-\frac{1}{8}$ correct choice

b. $-\frac{1}{16}$

c. 0

d. $\frac{1}{16}$

e. $\frac{1}{8}$

Solution: We use integration by parts with $\begin{array}{ll} u = x & dv = \cos(4x)\,dx \\ du = dx & v = \frac{1}{4}\sin(4x) \end{array}$. Then:

$$\int_0^{\pi/4} x\cos(4x)\,dx = \frac{x}{4}\sin(4x) - \frac{1}{4}\int \sin(4x)\,dx = \left[\frac{x}{4}\sin(4x) + \frac{1}{16}\cos(4x)\right]_0^{\pi/4}$$
$$= \left[\frac{\pi}{16}\sin(\pi) + \frac{1}{16}\cos(\pi)\right] - \left[\frac{0}{4}\sin(0) + \frac{1}{16}\cos(0)\right] = -\frac{1}{16} - \frac{1}{16} = -\frac{1}{8}$$

2. (5 pts) Compute $\int_0^{\pi/4} \tan^{5/2}x\sec^2x\,dx$.

a. $\frac{2}{3}$

b. $\frac{2}{5}$

c. $\frac{2}{7}$ correct choice

d. $\frac{2}{5}(2^{5/2}-1)$

e. $\frac{2}{7}(2^{5/2}-1)$

Solution: $u = \tan x \qquad du = \sec^2x\,dx.\qquad \int_0^{\pi/4} \tan^{5/2}x\sec^2x\,dx = \int_0^1 u^{5/2}\,du = \left[\frac{2u^{7/2}}{7}\right]_0^1 = \frac{2}{7}$

3. (1 pts) In computing the integral $\int_0^{\pi/4} \tan^{5/2}x\sec^2x\,dx$, you used the formula (identity):

a. $\frac{d}{dx}\tan x = \sec^2x$ correct choice

b. $\frac{d}{dx}\sec x = \sec x\tan x$

c. $\tan^2x + 1 = \sec^2x$

d. $\int \tan x\,dx = -\ln|\cos x| + C$

e. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$

4. (5 pts) Compute $\int \frac{3x^5}{\sqrt{x^3+8}}\,dx$.

a. $\frac{2(x^3+8)^{3/2}}{3} - 16(x^3+8)^{1/2} + C$ correct choice

b. $\frac{2(x^3+8)^{5/2}}{5} - \frac{16(x^3+8)^{3/2}}{3} + C$

c. $\frac{2(x^3+8)^{3/2}}{3} + 16(x^3+8)^{1/2} + C$

d. $\frac{2(x^3+8)^{5/2}}{5} + \frac{16(x^3+8)^{3/2}}{3} + C$

e. $\frac{2u^{3/2}}{3} + 16u^{3/2} + C$

f. $\frac{2u^{5/2}}{5} + \frac{16u^{3/2}}{3} + C$

g. $\frac{2u^{3/2}}{3} + 16u^{3/2} + C$

h. $\frac{2u^{3/2}}{3} + 16u^{3/2} + C$

Solution: We make the substitution $u = x^3 + 8$. The $du = 3x^2\,dx$ and $x^3 = u - 8$. So:

$$\int \frac{3x^5}{\sqrt{x^3+8}}\,dx = \int \frac{u-8}{\sqrt{u}}\,du = \int (u^{1/2} - 8u^{-1/2})\,du = \frac{2u^{3/2}}{3} - 16u^{1/2} + C = \frac{2(x^3+8)^{3/2}}{3} - 16(x^3+8)^{1/2} + C$$

5. (5 pts) Compute $\int x^2 \ln|x|\,dx$.

a. $\frac{x^3}{3}\ln|x| + \frac{x^4}{12} + C$

b. $\frac{x^3}{3}\ln|x| - \frac{x^4}{12} + C$

c. $\frac{x^3}{3}\ln|x| + \frac{x^2}{6} + C$

d. $\frac{x^3}{3}\ln|x| - \frac{x^2}{6} + C$

e. $\frac{x^3}{3}\ln|x| + \frac{x^3}{9} + C$

f. $\frac{x^3}{3}\ln|x| - \frac{x^3}{9} + C$ correct choice

Solution: We use integration by parts with $\begin{array}{ll} u = \ln|x| & dv = x^2\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^3}{3} \end{array}$. Then:

$$\int x^2 \ln|x|\,dx = \frac{x^3}{3}\ln|x| - \frac{1}{3}\int x^3 \frac{1}{x}\,dx = \frac{x^3}{3}\ln|x| - \frac{x^3}{9} + C$$

6. (1 pts) In computing the integral $\int x^2 \ln|x|\,dx$, you used:

a. the substitution $u = x^2$

b. the substitution $u = \ln|x|$

c. integration by parts with $u = x^2$

d. integration by parts with $u = \ln|x|$ correct choice

e. $\int \ln x\,dx = x\ln x - x + C$

f. $\int \ln x\,dx = x\ln x + x + C$

7. (5 pts) Let $A(x)$ be the area under the graph of the function $y = f(x)$ above the x-axis between $x = 2$ and a variable point x. If $A(x) = x^4 - 16$, then $f(x) =$

a. 0

b. $\frac{x^5}{5} - 16x + \frac{128}{5}$

c. $4x^3 - 32$

d. $\frac{x^5}{5} - 16x$

e. $4x^3$ correct choice

Solution: $A(x) = \int_2^x f(t)\,dt \qquad \frac{dA}{dx} = f(x) \qquad f(x) = \frac{dA}{dx} = 4x^3$

8. (5 pts) Compute $\int_0^{2/3} (4 - 9x^2)^{3/2}\,dx$

a. π correct choice

b. 2π

c. $\frac{2}{3}\pi$

d. $\frac{4}{3}\pi$

e. $\frac{5}{3}\pi$

Solution: Let $3x = 2\sin\theta$. Then $dx = \frac{2}{3}\cos\theta\,d\theta$ and

$$\int_0^{2/3} (4 - 9x^2)^{3/2}\,dx = \int_0^{\pi/2} (4 - 4\sin^2\theta)^{3/2}\,\frac{2}{3}\cos\theta\,d\theta = \frac{16}{3}\int_0^{\pi/2} (1 - \sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$= \frac{16}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{16}{3}\int_0^{\pi/2} \left(\frac{1 + \cos(2\theta)}{2}\right)^2 d\theta = \frac{4}{3}\int_0^{\pi/2} (1 + 2\cos(2\theta) + \cos^2(2\theta))\,d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2} \left(1 + 2\cos(2\theta) + \frac{1 + \cos(4\theta)}{2}\right) d\theta = \frac{4}{3}\left[\theta + \sin(2\theta) + \frac{1}{2}\left(\theta + \frac{\sin(4\theta)}{4}\right)\right]_0^{\pi/2}$$

$$= \frac{4}{3}\left(\frac{\pi}{2} + \frac{1}{2}\frac{\pi}{2}\right) = \pi$$

9. (1 pts) In computing the integral $\int_0^{2/3} (4 - 9x^2)^{3/2}\,dx$, you used:

a. the substitution $x = \frac{2}{3}\tan\theta$

b. the substitution $x = \frac{2}{3}\sin\theta$ correct choice

c. the substitution $x = \frac{2}{3}\sec\theta$

d. the substitution $x = \frac{3}{2}\tan\theta$

e. the substitution $x = \frac{3}{2}\sin\theta$

f. the substitution $x = \frac{3}{2}\sec\theta$

10. (1 pts) In computing the integral $\int_0^{2/3} (4 - 9x^2)^{3/2}\,dx$, you used the identity:

a. $\sin^2 A = \frac{1 + \cos(2A)}{2}$

b. $\sin^2 A = \frac{1 - \cos(2A)}{2}$

c. $\cos^2 A = \frac{1 + \cos(2A)}{2}$ correct choice

d. $\cos^2 A = \frac{1 - \cos(2A)}{2}$

11. (5 pts) Find the length of the parametric curve $x = t^2$ and $y = \frac{1}{3}t^3 - t$ for $0 \le t \le 3$.

a. 3
b. 6
c. 9
d. 12 correct choice
e. 16

Solution: $\frac{dx}{dt} = 2t \qquad \frac{dy}{dt} = t^2 - 1$

$$L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 \sqrt{(2t)^2 + (t^2-1)^2}\,dt = \int_0^3 \sqrt{4t^2 + (t^4 - 2t^2 + 1)}\,dt$$
$$= \int_0^3 \sqrt{t^4 + 2t^2 + 1}\,dt = \int_0^3 \sqrt{(t^2+1)^2}\,dt = \int_0^3 (t^2+1)\,dt = \left[\frac{1}{3}t^3 + t\right]_0^3 = 9 + 3 = 12$$

12. (5 pts) The curve $(x,y) = (\theta, \cosh\theta)$ for $0 \le \theta \le 1$ is rotated about the x-axis. Find the surface area swept out. Note:

$$\cosh^2\theta - \sinh^2\theta = 1 \qquad \cosh^2\theta = \frac{1+\cosh 2\theta}{2} \qquad \sinh^2\theta = \frac{\cosh 2\theta - 1}{2}$$

a. $\pi\left(1 - \frac{\cosh 2}{2}\right)$
b. $\pi\left(1 - \frac{\sinh 2}{2}\right)$
c. $2\pi\left(1 - \frac{\cosh 2}{2}\right)$
d. $2\pi\left(1 - \frac{\sinh 2}{2}\right)$
e. $\pi\left(1 + \frac{\cosh 2}{2}\right)$
f. $\pi\left(1 + \frac{\sinh 2}{2}\right)$ correct choice
g. $2\pi\left(1 + \frac{\cosh 2}{2}\right)$
h. $2\pi\left(1 + \frac{\sinh 2}{2}\right)$

Solution: $\frac{dx}{d\theta} = 1 \qquad \frac{dy}{d\theta} = \sinh\theta$. The radius is $r = y = \cosh\theta$. So the surface area is:

$$A = \int_0^1 2\pi r \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta = \int_0^1 2\pi\cosh\theta\sqrt{1 + (\sinh\theta)^2}\,d\theta = 2\pi\int_0^1 \cosh\theta\sqrt{\cosh^2\theta}\,d\theta$$
$$= 2\pi\int_0^1 \cosh^2\theta\,d\theta = 2\pi\int_0^1 \frac{1+\cosh 2\theta}{2}\,d\theta = \pi\left[\theta + \frac{\sinh 2\theta}{2}\right]_0^1 = \pi\left(1 + \frac{\sinh 2}{2}\right)$$

13. (5 pts) A rocket takes off from rest $(v(0) = 0)$ at the ground $(y(0) = 0)$ and has acceleration $a(t) = 40e^{-2t}$. Find its height at $t = 2$.

a. $10e^{-4}$
b. $40e^{-4}$
c. $160e^{-4}$
d. $10e^{-4} + 30$ correct choice
e. $10e^{-4} + 10$

Solution: $\frac{dv}{dt} = a = 40e^{-2t} \qquad v = -20e^{-2t} + C \qquad v(0) = -20 + C = 0 \quad \Rightarrow \quad C = 20$

$\frac{dy}{dt} = v = -20e^{-2t} + 20 \qquad y = 10e^{-2t} + 20t + K \qquad y(0) = 10 + K = 0 \quad \Rightarrow \quad K = -10$

$y = 10e^{-2t} + 20t - 10 \qquad y(2) = 10e^{-2\cdot 2} + 20\cdot 2 - 10 = 10e^{-4} + 30$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (15 pts) A bar between $x = 2$ and $x = 4$ has linear density $\delta = \frac{1}{x^3}$ g/cm.

a. Find the total mass of the bar.

Solution: $M = \int \delta\, dx = \int_2^4 \frac{1}{x^3}\, dx = \left[-\frac{1}{2x^2}\right]_2^4 = -\frac{1}{32} + \frac{1}{8} = \frac{3}{32}$

b. Find the center of mass of the bar.

Solution: $M_1 = \int x\delta\, dx = \int_2^4 \frac{x}{x^3}\, dx = \int_2^4 \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_2^4 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}$

$\bar{x} = \frac{M_1}{M} = \frac{1}{4}\frac{32}{3} = \frac{8}{3}$

15. (20 pts) Compute $\int e^{2x} \sin 3x\, dx$.

The answer has the form $Ae^{2x}\sin 3x + Be^{2x}\cos 3x + C$. Then $A =$ ____ and $B =$ ____.

Solution: Use parts with $u = \sin 3x$, $dv = e^{2x}\, dx$, $du = 3\cos 3x\, dx$, $v = \frac{1}{2}e^{2x}$. Then

$$I = \int e^{2x}\sin 3x\, dx = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\int e^{2x}\cos 3x\, dx$$

Next use parts with $u = \cos 3x$, $dv = e^{2x}\, dx$, $du = -3\sin 3x\, dx$, $v = \frac{1}{2}e^{2x}$

$$I = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{2}\left[\frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\sin 3x\, dx\right] = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x - \frac{9}{4}I$$

$$I + \frac{9}{4}I = \frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x$$

$$I = \frac{4}{13}\left(\frac{1}{2}e^{2x}\sin 3x - \frac{3}{4}e^{2x}\cos 3x\right) + C = \frac{2}{13}e^{2x}\sin 3x - \frac{3}{13}e^{2x}\cos 3x + C$$

16. (20 pts) Compute $\int \frac{\sqrt{x^2 - 9}}{x}\, dx$.

The answer has the form $A(x^2 - 9)^{3/2} + B\sqrt{x^2 - 9} + C\,\mathrm{arcsec}\,\frac{x}{3} + K$.

Then $A =$ ____, $B =$ ____ and $C =$ ____.

Solution: Let $x = 3\sec\theta$. Then $dx = 3\sec\theta\tan\theta\, d\theta$. So

$$I = \int \frac{\sqrt{x^2 - 9}}{x}\, dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{3\sec^2\theta} 3\sec\theta\tan\theta\, d\theta = 3\int \tan^2\theta\, d\theta = 3\int \sec^2\theta - 1\, d\theta$$

$$= 3\tan\theta - 3\theta + K$$

Draw a triangle with hypotenous x, adjacent side 3 and opposite side $\sqrt{x^2 - 9}$. Then

$$I = 3\frac{\sqrt{x^2 - 9}}{3} - 3\,\mathrm{arcsec}\,\frac{x}{3} + K = \sqrt{x^2 - 9} - 3\,\mathrm{arcsec}\,\frac{x}{3} + K$$