Name____

1-11	/55	13	/15
12	/20	14	/15
		Total	/105

MATH 172H Exam 3 Spring 2021

Sections 200 Solutions P. Yasskin

Multiple Choice and Short Answer: (Points indicated.)

1. (5 pts) Compute $\lim_{n\to\infty}\left(\sqrt{n^2-4n+3}-\sqrt{n^2+5n-2}\right)$.

a. 0

b. -9

c. $-\frac{9}{2}$ correct choice

d. $\frac{9}{2}$

e. 9

Solution: Multiply and divide by the conjugate

$$\lim_{n\to\infty}\left(\sqrt{n^2-4n+3}-\sqrt{n^2+5n-2}\right)=\lim_{n\to\infty}\left(\sqrt{n^2-4n+3}-\sqrt{n^2+5n-2}\right)\frac{\sqrt{n^2-4n+3}+\sqrt{n^2+5n-2}}{\sqrt{n^2-4n+3}+\sqrt{n^2+5n-2}}$$

$$=\lim_{n\to\infty}\frac{(n^2-4n+3)-(n^2+5n-2)}{\sqrt{n^2-4n+3}+\sqrt{n^2+5n-2}}=\lim_{n\to\infty}\frac{-9n+5}{\sqrt{n^2-4n+3}+\sqrt{n^2+5n-2}}=-\frac{9}{2}$$

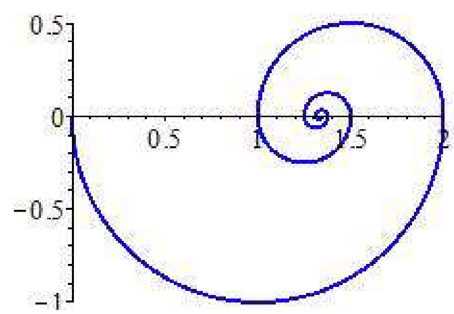
2. (5 pts) Compute $L=\lim_{n\to\infty}n^{1/n}$ (Type infinity for ∞, pi for π and e for e.)

$L=$ __1__

Solution: Let $L=\lim_{n\to\infty}n^{1/n}$. Using l'Hospital's rule,

$$\ln L=\lim_{n\to\infty}\ln n^{1/n}=\lim_{n\to\infty}\frac{\ln n}{n}\overset{l'H}{=}\lim_{n\to\infty}\frac{\frac{1}{n}}{1}=0 \qquad L=e^{\ln L}=e^0=1$$

3. (5 pts) The spiral at the right is made from an infinite number of semicircles whose centers are all on the x-axis. The first semicircle has radius $r_1=1$. The radius of each subsequent semicircle is half of the radius of the previous semicircle. Find the total length of the spiral. (Type infinity for ∞, pi for π and e for e.)



$L=$ __2π__

Solution: The radii are $r_1=1,\quad r_2=\frac{1}{2},\quad\cdots,\quad r_n=\frac{1}{2^{n-1}}$.

The lengths of the semicircles are $L_1=\pi,\quad L_2=\frac{\pi}{2},\quad\cdots,\quad L_n=\frac{\pi}{2^{n-1}}$.

The total length is $L=\sum_{n=1}^{\infty}L_n=\sum_{n=1}^{\infty}\frac{\pi}{2^{n-1}}=\frac{\pi}{1-\frac{1}{2}}=2\pi$

4. (5 pts) Compute $\displaystyle\sum_{n=3}^{\infty}\left(\frac{\sqrt{n}}{\sqrt{n+1}}-\frac{\sqrt{n+1}}{\sqrt{n+2}}\right)$

a. $\dfrac{\sqrt{3}}{2}$

b. $\dfrac{2-\sqrt{3}}{2}$

c. 0

d. $\dfrac{\sqrt{3}-2}{2}$ correct choice

e. $\dfrac{-\sqrt{3}}{2}$

Solution: The k^{th} partial sum is

$$S_k=\sum_{n=3}^{k}\left(\frac{\sqrt{n}}{\sqrt{n+1}}-\frac{\sqrt{n+1}}{\sqrt{n+2}}\right)=\left(\frac{\sqrt{3}}{\sqrt{4}}-\frac{\sqrt{4}}{\sqrt{5}}\right)+\left(\frac{\sqrt{4}}{\sqrt{5}}-\frac{\sqrt{5}}{\sqrt{6}}\right)+\cdots+\left(\frac{\sqrt{k}}{\sqrt{k+1}}-\frac{\sqrt{k+1}}{\sqrt{k+2}}\right)$$

$$=\frac{\sqrt{3}}{2}-\frac{\sqrt{k+1}}{\sqrt{k+2}} \qquad S=\lim_{k\to\infty}\left(\frac{\sqrt{3}}{2}-\frac{\sqrt{k+1}}{\sqrt{k+2}}\right)=\frac{\sqrt{3}}{2}-1=\frac{\sqrt{3}-2}{2}$$

5. (5 pts) Which of the following are correct about the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+\sqrt{n}}$?

Answer all that are correct.

Scoring: Grade $=\dfrac{\text{\# answered correctly}}{\text{\# correct answers}}\cdot 5-\text{\# answered incorrectly}$

a. diverges by the n^{th} Term Divergence Test

b. diverges by the Simple Comparison Test comparing to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

c. diverges by the Limit Comparison Test comparing to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

d. converges because it is a p-series

e. converges by the Simple Comparison Test comparing to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ correct choice

f. converges by the Limit Comparison Test comparing to $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ correct choice

g. converges by the Ratio Test

Solution: $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series since $p=2>1$.

$\dfrac{1}{n^2+\sqrt{n}}<\dfrac{1}{n^2}$ So it converges by the Simple Comparison Test

$L=\displaystyle\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{1}{n^2+\sqrt{n}}\frac{n^2}{1}=1 \qquad 0<L<\infty$ So it converges by the Limit Comparison Test.

Since it converges, it cannot diverge. It is not a p-series. The Ratio Test fails because $\rho=1$.

6. (5 pts) Find a power series about $x = 0$ for $f(x) = \dfrac{4x^3}{1-x^2}$.

a. $\sum_{n=0}^{\infty}(4x^3)^{2n}$

b. $\sum_{n=0}^{\infty} 8nx^{2n+3}$

c. $\sum_{n=0}^{\infty} 4x^{2n+3}$ correct choice

d. $\sum_{n=0}^{\infty} 4x^{2(n+3)}$

e. $\sum_{n=0}^{\infty} 4nx^{2n+3}$

f. $\sum_{n=0}^{\infty} 4nx^{2(n+3)}$

Solution: $\dfrac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n}$ $\dfrac{4x^3}{1-x^2} = \sum_{n=0}^{\infty} 4x^{2n+3}$

7. (5 pts) Find a power series about $x = 0$ for $f(x) = \dfrac{2x}{(1-x^2)^2}$.

a. $\sum_{n=0}^{\infty} 2nx^{2n-1}$ correct choice

b. $\sum_{n=0}^{\infty} 2x^{2n-1}$

c. $\sum_{n=0}^{\infty} 2nx^{2n+1}$

d. $\sum_{n=0}^{\infty} 2x^{2n+1}$

e. $\sum_{n=0}^{\infty} 4n^3x^{2n-1}$

f. $\sum_{n=0}^{\infty} 4n^3x^{2n+1}$

Solution: $\dfrac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n = \sum_{n=0}^{\infty} x^{2n}$

$$\frac{d}{dx}\frac{1}{1-x^2} = \frac{-1(-2x)}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2} = \sum_{n=0}^{\infty} 2nx^{2n-1}$$

8. (5 pts) Find the Taylor series for $f(x) = \frac{1}{x}$ about $x = 2$.

a. $\sum_{n=0}^{\infty} \frac{1}{2^n} x^n$

b. $\sum_{n=0}^{\infty} \frac{1}{2^n} (x-2)^n$

c. $\sum_{n=0}^{\infty} \frac{n!}{2^n} x^n$

d. $\sum_{n=0}^{\infty} \frac{n!}{2^n} (x-2)^n$

e. $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n$

f. $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} (x-2)^n$

g. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} x^n$

h. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (x-2)^n$

i. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} x^n$

j. $\sum_{n=0}^{\infty} \frac{(-1)^n n!}{2^n} (x-2)^n$

k. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n$

l. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (x-2)^n$ correct choice

Solution: We make a table of the function and several derivatives and evaluate at $x = 2$. We then generalize to the n^{th} derivative:

$$f(x) = \frac{1}{x} \qquad f(2) = \frac{1}{2}$$

$$f'(x) = -\frac{1}{x^2} \qquad f'(2) = -\frac{1}{2^2}$$

$$f''(x) = \frac{2}{x^3} \qquad f''(2) = \frac{2}{2^3}$$

$$f'''(x) = -\frac{3!}{x^4} \qquad f'''(2) = -\frac{3!}{2^4}$$

$$f^{(n)}(x) = (-1)^n \frac{n!}{x^{n+1}} \qquad f^{(n)}(2) = (-1)^n \frac{n!}{2^{n+1}}$$

Finally, we plug into the Taylor series:

$$Tf = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \qquad \frac{1}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \frac{n!}{2^{n+1}}}{n!}(x-2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$$

9. (5 pts) Use the 3^{rd} degree Taylor polynomial for $\sin(x)$ centered at $x = 0$ to approximate $\sin(0.3)$.

a. $.3$

b. $.309$

c. $.291$

d. $.3045$

e. $.2955$ correct choice

Solution: $\sin(x) \approx x - \frac{x^3}{3!}$ $\qquad \sin(.3) \approx .3 - \frac{(.3)^3}{6} = .3 - .0045 = .2955$

10. (5 pts) Compute $S = \sum_{n=0}^{\infty} \frac{1}{2^n n!}$

a. $\sin(2)$

b. $\sin\left(\frac{1}{2}\right)$

c. $\frac{\sin(1)}{2}$

d. e^2

e. $\sqrt{e}$ correct choice

f. $\frac{e}{2}$

g. $\cos(2)$

h. $\cos\left(\frac{1}{2}\right)$

i. $\frac{\cos(1)}{2}$

j. -1

k. 2

l. ∞

Solution: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ Here $x = \frac{1}{2}$. So $\sum_{n=0}^{\infty} \frac{1}{2^n n!} = e^{1/2} = \sqrt{e}$

11. (5 pts) Compute $L = \lim_{x\to\infty} \frac{1 - \cos(2x)}{x^2}$

$L =$ __2__

Solution: $\cos(u) = 1 - \frac{u^2}{2} + \frac{u^4}{4!} \cdots \qquad \cos(2x) = 1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots$

$$\lim_{x\to\infty} \frac{1 - \cos(2x)}{x^2} = \lim_{x\to\infty} \frac{1 - \left[1 - \frac{4x^2}{2} + \frac{16x^4}{4!} + \cdots\right]}{x^2} = \lim_{x\to\infty} \frac{\frac{4x^2}{2} - \frac{16x^4}{4!} + \cdots}{x^2}$$

$$= \lim_{x\to\infty}\left(\frac{4}{2} - \frac{16x^2}{4!} + \cdots\right) = 2$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (20 pts) Work Out Problem

For each power series, find the radius and interval of convergence.
Give complete explanations. (Type infinity for ∞.)

a. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+1)}(x-3)^n$

$R = $ __2__ $\qquad I = $ ____$(1,5]$____

Solution: We apply the ratio test:

$$a_n = \frac{(-1)^n(x-3)^n}{2^n(n+1)} \qquad a_{n+1} = \frac{(-1)^{n+1}(x-3)^{n+1}}{2^{n+1}(n+2)}$$

$$\rho = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x-3|^{n+1}}{2^{n+1}(n+2)}\frac{2^n(n+1)}{|x-3|^n} = \frac{|x-3|}{2}\lim_{n\to\infty}\frac{n+1}{n+2} = \frac{|x-3|}{2} < 1$$

Converges when $|x-3| < 2$ So $R = 2$. The open interval of convergence is $(1,5)$.

We check endpoints:

$x = 1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^n}(-2)^n = \sum_{n=0}^{\infty}\frac{1}{(n+1)}$ divergent harmonic series

$x = 5$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)2^n}(2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)}$ convergent alternating harmonic series

So the interval of convergence is $(1,5]$.

b. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n(n+1)!}(x-3)^n$

$R = $ __∞__ $\qquad I = $ ____$(-\infty,\infty)$____

Solution: We apply the ratio test:

$$a_n = \frac{(-1)^n(x-3)^n}{2^n(n+1)!} \qquad a_{n+1} = \frac{(-1)^{n+1}(x-3)^{n+1}}{2^{n+1}(n+2)!}$$

$$\rho = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{|x-3|^{n+1}}{2^{n+1}(n+2)!}\frac{2^n(n+1)!}{|x-3|^n} = \frac{|x-3|}{2}\lim_{n\to\infty}\frac{1}{n+2} = 0 < 1$$

Converges for all x. $R = \infty$. So the interval of convergence is $(-\infty,\infty)$.

13. (15 pts) Work Out Problem

Consider the sequence given by the recursion relation $a_{n+1} = 2\sqrt{a_n}$ starting from $a_1 = 1$.
Does the sequence have a limit? If so, find the limit. If not, enter divergent.
Be sure to use sentences, name the theorem you use and prove all statements.

$\lim_{n\to\infty} a_n =$ ____

Solution: The first 3 terms are: $a_1 = 1$, $a_2 = 2\sqrt{1} = 2$, $a_3 = 2\sqrt{2}$ This appears to be increasing.
Assuming the limit exists, let $L = \lim_{n\to\infty} a_n$. Then $L = 2\sqrt{L}$ or $L^2 = 4L$ or $L = 0, 4$.
So if a limit exists, it must be 0 or 4.

We use induction to prove the sequence is increasing and bounded above by 4, i.e. $a_n < a_{n+1} < 4$.

Initialization Step: $a_1 < a_2 < 4$ because $1 < 2 < 4$.

Induction Step: Assume $a_k < a_{k+1} < 4$. Prove $a_{k+1} < a_{k+2} < 4$.

Proof:

$$a_k < a_{k+1} < 4 \quad \Rightarrow \quad \sqrt{a_k} < \sqrt{a_{k+1}} < \sqrt{4} = 2 \quad \Rightarrow \quad 2\sqrt{a_k} < 2\sqrt{a_{k+1}} < 4 \quad \Rightarrow \quad a_{k+1} < a_{k+2} < 4$$

By the Bounded Monotonic Sequence Theorem, since the function is increasing and bounded above by 4, it has a limit, and $\lim_{n\to\infty} a_n = 4$.

14. (15 pts) Work Out Problem

Give a complete explantion as to why the series $\sum_{n=2}^{\infty} \frac{(-1)^n (n+1)}{n^2 + \sqrt{n}}$ is absolutely convergent, conditionally convergent or divergent.

a. absolutely convergent

b. conditionally convergent

c. divergent

Solution: The related absolute series is $\sum_{n=2}^{\infty} \frac{n+1}{n^2 + \sqrt{n}}$. We will compare to $\sum_{n=1}^{\infty} \frac{1}{n}$ which is the divergent harmonic series. We cannot use the Simple Comparison Test because there is no good inequality. So we apply the Limit Comparison Test. $L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^2 + \sqrt{n}} \frac{n}{1} = 1$ Since $0 < L < \infty$, the absolute series also diverges.

We test the original series by the Alternating Series Test. The absolute value of the terms is $b_n = \frac{n+1}{n^2 + \sqrt{n}}$ which is positive and decreasing and $\lim_{n\to\infty} \frac{n+1}{n^2 + \sqrt{n}} = 0$. So the original series converges and is conditionally convergent.