1-9	/54	12	/10
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Name________________________________

MATH 172 Honors Exam 1 Spring 2022

Sections 200 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit. Circle your answers.)

1. Find the area between $y = x^3 - 4x$ and $y = 5x$. (Circle your answer.)

a. 3 d. 9 g. 27 j. 81

b. $\frac{3}{2}$ e. $\frac{9}{2}$ h. $\frac{27}{2}$ k. $\boxed{\frac{81}{2}}$

c. $\frac{3}{4}$ f. $\frac{9}{4}$ i. $\frac{27}{4}$ l. $\frac{81}{4}$

Solution: The curves intersect when $x^3 - 4x = 5x$ or $0 = x^3 - 9x = x(x-3)(x+3)$ or $x = -3, 0, 3$.

$$A = \int_{-3}^{0} (x^3 - 4x - 5x)\,dx + \int_{0}^{3} (5x - x^3 + 4x)\,dx = 2\int_{0}^{3} (9x - x^3)\,dx$$

$$= 2\left[9\frac{x^2}{2} - \frac{x^4}{4}\right]_0^3 = 2\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81}{2}$$

2. Find the average value of $g(x) = x^5 - x^2$ on $[0,3]$. (Circle your answer.)

a. 5 d. 25 g. 75 j. 225

b. $\frac{5}{2}$ e. $\frac{25}{2}$ h. $\boxed{\frac{75}{2}}$ k. $\frac{225}{2}$

c. $\frac{5}{4}$ f. $\frac{25}{4}$ i. $\frac{75}{4}$ l. $\frac{225}{4}$

Solution: $g_{\text{ave}} = \frac{1}{b-a}\int_a^b g(x)\,dx = \frac{1}{3}\int_0^3 (x^5 - x^2)\,dx = \frac{1}{3}\left[\frac{x^6}{6} - \frac{x^3}{3}\right]_0^3$

$$= \frac{1}{3}\left(\frac{3^6}{6} - \frac{3^3}{3}\right) = \frac{3^4}{2} - 3 = \frac{81}{2} - \frac{6}{2} = \frac{75}{2}$$

3. Compute $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\arctan x}{1+x^2}\,dx$. (Circle your answer.)

a. $\frac{\pi}{9}$ d. $\frac{\pi}{24}$ g. $\frac{\pi^2}{9}$ j. $\boxed{\frac{\pi^2}{24}}$

b. $\frac{\pi}{12}$ e. $\frac{\pi}{36}$ h. $\frac{\pi^2}{12}$ k. $\frac{\pi^2}{36}$

c. $\frac{\pi}{18}$ f. $\frac{\pi}{72}$ i. $\frac{\pi^2}{18}$ l. $\frac{\pi^2}{72}$

Solution: Let $u = \arctan x$. Then $du = \frac{1}{1+x^2}\,dx$.

Further, $\arctan\sqrt{3} = \frac{\pi}{3}$ and $\arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. So

$$\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{\arctan x}{1+x^2}\,dx = \int_{\pi/6}^{\pi/3} u\,du = \left[\frac{u^2}{2}\right]_{\pi/6}^{\pi/3} = \frac{1}{2}\left(\frac{\pi^2}{9} - \frac{\pi^2}{36}\right) = \frac{\pi^2}{24}$$

4. Compute $\int_0^{\pi/4} \tan^2 x \sec^4 x\, dx$. (Circle your answer.)

a. $\boxed{\frac{8}{15}}$ d. $\frac{4}{15}$ g. $\frac{2}{15}$ j. $\frac{1}{15}$

b. $\frac{8}{5}$ e. $\frac{4}{5}$ h. $\frac{2}{5}$ k. $\frac{1}{5}$

c. $\frac{8}{3}$ f. $\frac{4}{3}$ i. $\frac{2}{3}$ l. $\frac{1}{3}$

Solution: Let $u = \tan x$. Then $du = \sec^2 x\, dx$ and $\sec^2 x = \tan^2 x + 1 = u^2 + 1$.

$$\int_0^{\pi/4} \tan^2 x \sec^4 x\, dx = \int_0^1 u^2(u^2+1)\, du = \left[\frac{u^5}{5} + \frac{u^3}{3} \right]_0^1 = \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$$

5. Compute $\int_0^{\pi/3} \tan\theta \sec^3\theta\, d\theta$. (Circle your answer.)

a. $\frac{3}{5}$ d. $\frac{3}{7}$ g. $\frac{3}{5}\pi$ j. $\frac{3}{7}\pi$

b. $\frac{5}{3}$ e. $\frac{5}{7}$ h. $\frac{5}{3}\pi$ k. $\frac{5}{7}\pi$

c. $\boxed{\frac{7}{3}}$ f. $\frac{7}{5}$ i. $\frac{7}{3}\pi$ l. $\frac{7}{5}\pi$

Solution: Let $u = \sec\theta$. Then $du = \sec\theta\tan\theta\, d\theta$.

Further, $\sec\frac{\pi}{3} = 2$ and $\sec 0 = 1$. So

$$\int_0^{\pi/3} \tan\theta \sec^3\theta\, d\theta = \int_1^2 u^2\, du = \left[\frac{u^3}{3} \right]_1^2 = \left(\frac{8}{3} - \frac{1}{3} \right) = \frac{7}{3}$$

6. Compute $\int_0^{\pi} \sin^4\theta \cos^4\theta\, d\theta$. (Circle your answer.)

a. $\frac{1}{32}\pi$ d. $\frac{3}{32}\pi$ g. $\frac{5}{32}\pi$ j. $\frac{7}{32}\pi$

b. $\frac{1}{64}\pi$ e. $\frac{3}{64}\pi$ h. $\frac{5}{64}\pi$ k. $\frac{7}{64}\pi$

c. $\frac{1}{128}\pi$ f. $\boxed{\frac{3}{128}\pi}$ i. $\frac{5}{128}\pi$ l. $\frac{7}{128}\pi$

Solution: We use $\sin\theta\cos\theta = \dfrac{\sin(2\theta)}{2}$, $\sin^2(2\theta) = \dfrac{1-\cos(4\theta)}{2}$ and $\cos^2(4\theta) = \dfrac{1+\cos(8\theta)}{2}$.

$$\int_0^{\pi} \sin^4\theta\cos^4\theta\, d\theta = \int_0^{\pi} \left(\frac{\sin(2\theta)}{2} \right)^4 d\theta = \frac{1}{16}\int_0^{\pi} \left(\frac{1-\cos(4\theta)}{2} \right)^2 d\theta = \frac{1}{64}\int_0^{\pi} (1 - 2\cos(4\theta) + \cos^2(4\theta))\, d\theta$$

$$= \frac{1}{64}\int_0^{\pi} \left(1 - 2\cos(4\theta) + \frac{1+\cos(8\theta)}{2} \right) d\theta = \frac{1}{64}\left[\theta - \frac{\sin(4\theta)}{2} + \frac{1}{2}\left(\theta + \frac{\sin(8\theta)}{8} \right) \right]_0^{\pi}$$

$$= \frac{1}{64}\left[\pi + \frac{1}{2}\pi \right] = \frac{3}{128}\pi$$

7. What trig substitution should you make to do the integral $\int_0^{1/2} \frac{x^2}{\sqrt{4-9x^2}}\,dx$? (Circle your answer.)

a. $x = \frac{2}{3}\tan\theta$ d. $x = \frac{3}{2}\tan\theta$

b. $\boxed{x = \frac{2}{3}\sin\theta}$ e. $x = \frac{3}{2}\sin\theta$

c. $x = \frac{2}{3}\sec\theta$ f. $x = \frac{3}{2}\sec\theta$

Solution: The $\frac{2}{3}$ is needed so the 4 factors out. The minus says we need a $\sin$ or $\sec$.
The substitution $x = \frac{2}{3}\sin\theta$ says $|x| \le \frac{2}{3}$ while $x = \frac{2}{3}\sec\theta$ says $|x| \ge \frac{2}{3}$.
The square root say $4 - 9x^2 \ge 0$ or $9x^2 \le 4$ or $|x| \le \frac{2}{3}$. So its a $\sin$ sub.

8. What trig substitution should you make to do the integral $\int_4^9 \frac{x^2}{4-9x^2}\,dx$? (Circle your answer.)

a. $x = \frac{2}{3}\tan\theta$ d. $x = \frac{3}{2}\tan\theta$

b. $x = \frac{2}{3}\sin\theta$ e. $x = \frac{3}{2}\sin\theta$

c. $\boxed{x = \frac{2}{3}\sec\theta}$ f. $x = \frac{3}{2}\sec\theta$

Solution: The $\frac{2}{3}$ is needed so the 4 factors out. The minus says we need a $\sin$ or $\sec$.
The substitution $x = \frac{2}{3}\sin\theta$ says $|x| \le \frac{2}{3}$ while $x = \frac{2}{3}\sec\theta$ says $|x| \ge \frac{2}{3}$.
The limits say $|x| \ge 4$. So its a $\sec$ sub.

9. Compute $\int \frac{1}{\sqrt{4x^2-1}}\,dx$ (Circle your answer. The $+C$ is understood.)

a. $\operatorname{arcsec}(2x) + \frac{1}{2}\ln\left|\frac{1}{2x} + \frac{1}{\sqrt{4x^2-1}}\right|$ d. $\frac{1}{2}\ln\left|\frac{1}{2x} + \frac{1}{\sqrt{4x^2-1}}\right|$ g. $-\frac{1}{2}\ln\left|\frac{1}{2x}\right|$

b. $\operatorname{arcsec}(2x) + \frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right|$ e. $\boxed{\frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right|}$ h. $\frac{1}{2}\ln\left|\frac{\sqrt{4x^2-1}}{2x}\right|$

c. $\operatorname{arcsec}(2x) + \frac{1}{2}\ln\left|\frac{2x + \sqrt{4x^2-1}}{2x}\right|$ f. $\frac{1}{2}\ln\left|\frac{2x + \sqrt{4x^2-1}}{2x}\right|$ i. $\frac{1}{4}\sqrt{4x^2-1}$

Solution: Let $2x = \sec\theta$. Then $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. So

$$I = \int \frac{1}{\sqrt{4x^2-1}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,d\theta = \frac{1}{2}\int \frac{1}{\tan\theta}\sec\theta\tan\theta\,d\theta = \frac{1}{2}\int \sec\theta\,d\theta$$
$$= \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

Draw a triangle with hypotenous $2x$, adjacent side 1 and opposite side $\sqrt{4x^2-1}$. So

$$I = \frac{1}{2}\ln\left|2x + \sqrt{4x^2-1}\right| + C$$

Work Out: (10 points each. Part credit possible. Show all work.)

10. Compute $\int e^{4x}\sin 3x\,dx$.

Solution: Use parts with $\begin{array}{ll} u = \sin 3x & dv = e^{4x}\,dx \\ du = 3\cos 3x\,dx & v = \frac{1}{4}e^{4x} \end{array}$. Then

$$I = \int e^{4x}\sin 3x\,dx = \frac{1}{4}e^{4x}\sin 3x - \frac{3}{4}\int e^{4x}\cos 3x\,dx$$

Next use parts with $\begin{array}{ll} u = \cos 3x & dv = e^{4x}\,d \\ du = -3\sin 3x\,dx & v = \frac{1}{4}e^{4x} \end{array}$

$$I = \frac{1}{4}e^{4x}\sin 3x - \frac{3}{4}\left[\frac{1}{4}e^{4x}\cos 3x + \frac{3}{4}\int e^{4x}\sin 3\,dx\right] = \frac{1}{4}e^{4x}\sin 3x - \frac{3}{16}e^{4x}\cos 3x - \frac{9}{16}I$$

$$I + \frac{9}{16}I = \frac{1}{4}e^{4x}\sin 3x - \frac{3}{16}e^{4x}\cos 3x$$

$$I = \frac{16}{25}\left(\frac{1}{4}e^{4x}\sin 3x - \frac{3}{16}e^{4x}\cos 3x\right) + C = \frac{4}{25}e^{4x}\sin 3x - \frac{3}{25}e^{4x}\cos 3x + C$$

11. A bar of length π m has linear density $\delta = \sin x$ kg/m where x is measured from one end.

a. Find the total mass of the bar.

Solution: $M = \int \delta\,dx = \int_0^\pi \sin x\,dx = \left[-\cos x\right]_0^\pi = (--1) - (-1) = 2$

b. Find the center of mass of the bar.

Solution: $M_1 = \int x\delta\,dx = \int_0^\pi x\sin x\,dx$ Use parts with $\begin{array}{ll} u = x & dv = \sin x\,dx \\ du = dx & v = -\cos x \end{array}$

$$M_1 = \left[-x\cos x + \int \cos x\,dx\right]_0^\pi = \left[-x\cos x + \sin x\right]_0^\pi = (-\pi\cos\pi + \sin\pi) = \pi$$

$$\bar{x} = \frac{M_1}{M} = \frac{\pi}{2}$$

12. Find the arc length of the $3D$ parametric curve $\vec{r}(t) = \left\langle t^3, \sqrt{\frac{3}{2}}\, t^2, t \right\rangle$ for $0 \le t \le 4$.

Solution: $\frac{dx}{dt} = 3t^2$ $\quad \frac{dy}{dt} = \sqrt{6}\, t$ $\quad \frac{dz}{dt} = 1$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{(3t^2)^2 + \left(\sqrt{6}\, t\right)^2 + (1)^2}\, dt$$

$$= \sqrt{9t^4 + 6t^2 + 1}\, dt = \sqrt{(3t^2 + 1)^2}\, dt = (3t^2 + 1)\, dt$$

$$L = \int ds = \int_0^4 (3t^2 + 1)\, dt = \left[t^3 + t\right]_0^4 = 64 + 4 = 68$$

13. The curve $y = 2\sqrt{x}$ for $0 \le x \le 3$ is rotated about the x-axis. Find the surface area.

Solution: $\frac{dy}{dx} = \frac{1}{\sqrt{x}}$. The radius is $r = y = 2\sqrt{x}$. So the surface area is:

$$A = \int 2\pi r \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 2\pi 2\sqrt{x}\, \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}\, dx = 4\pi \int_0^3 \sqrt{x}\, \sqrt{1 + \frac{1}{x}}\, dx$$

$$= 4\pi \int_0^3 \sqrt{x + 1}\, dx = 4\pi \left[\frac{2(x+1)^{3/2}}{3}\right]_0^3 = \frac{8\pi}{3}(4^{3/2} - 1) = \frac{56\pi}{3}$$

14. Consider the curve $y = f(x) = 64 - x^3$

a. Find the equation of the tangent line to $y = 64 - x^3$ at the general point $x = p$.

Solution: $f(x) = 64 - x^3$ $\quad f'(x) = -3x^2$ $\quad f(p) = 64 - p^3$ $\quad f'(p) = -3p^2$

The tangent line is $y = f(p) + f'(p)(x - p) = 64 - p^3 - 3p^2(x - p) = 64 + 2p^3 - 3p^2 x$

b. Find the area, $A(p)$, under this tangent line above the x-axis over the interval $[0,4]$.

Solution: The area under this tangent line is

$$A = \int_0^4 (64 + 2p^3 - 3p^2 x)\,dx = \left[64x + 2p^3 x - 3p^2 \frac{x^2}{2} \right]_0^4 = 256 + 8p^3 - 24p^2$$

c. Find the value of p for which this area, $A(p)$, is a minimum.
Be sure to use the second derivative test to check it is a minimum.

Solution: We find the critical points of $A(p)$:

$A' = 24p^2 - 48p = 24p(p - 2) = 0$ at $p = 0$ or 2

We compute the second derivative and test each critical point to see which is the minimum:

$A'' = 48p - 48$

$A''(0) = -48 < 0$ $\quad p = 0$ is the local maximum.

$A''(2) = 48 > 0$ $\quad p = 2$ is the local minimum.