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MATH 221 Exam 2 Version A Fall 2019

Section 505 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-9	/54	11	/20
10	/5	12	/25
		Total	/104

1. Find the equation of the plane tangent to $z = x^2y^2 + xy^2$ at the point $(x,y) = (1,2)$.

Its z -intercept is:

- a. $c = 20$
- b. $c = 8$
- c. $c = -8$
- d. $c = -16$
- e. $c = -20$ Correct Choice

Solution: $f(x,y) = x^2y^2 + xy^2$ $f_x(x,y) = 2xy^2 + y^2$ $f_y(x,y) = 2x^2y + 2xy$

$$f(1,2) = 1^2 \cdot 2^2 + 1 \cdot 2^2 = 8 \quad f_x(1,2) = 2 \cdot 1 \cdot 2^2 + 2^2 = 12 \quad f_y(1,2) = 2 \cdot 1^2 \cdot 2 + 2 \cdot 1 \cdot 2 = 8$$

Tangent plane: $z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 8 + 12(x-1) + 8(y-2)$

$$z = 12x + 8y - 20 \quad z\text{-intercept is } c = -20.$$

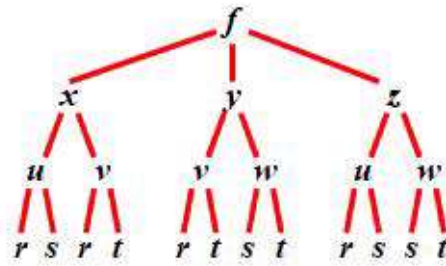
2. Use differentials to estimate the volume of metal needed to make a cylindrical tin can with lids if the radius is $r = 5$ cm and the height is $h = 6$ cm and the metal has thickness .02 cm?

- a. 200π
- b. 4π
- c. 105π
- d. 2.2π Correct Choice
- e. 2.6π

Solution: $V = \pi r^2 h$ $\Delta V \approx dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = 2\pi r h dr + \pi r^2 dh$

$$r = 5 \quad h = 6 \quad dr = .02 \quad dh = .04 \quad \Delta V \approx 2\pi(5)(6)(.02) + \pi(5)^2(.04) = 2.2\pi$$

3. At the right is a tree diagram showing f as a function of x , y and z which are functions of u , v and w which are functions of r , s and t as indicated.



Below are values of a bunch partial derivatives.
Use this information to compute $\frac{\partial f}{\partial s}$.

$$\begin{array}{lll} \frac{\partial f}{\partial x} = 2 & \frac{\partial f}{\partial y} = 3 & \frac{\partial f}{\partial z} = 4 \\ \frac{\partial x}{\partial u} = 5 & \frac{\partial x}{\partial v} = 6 & \frac{\partial y}{\partial v} = 7 & \frac{\partial y}{\partial w} = 8 & \frac{\partial z}{\partial u} = 9 & \frac{\partial z}{\partial w} = 10 \\ \frac{\partial u}{\partial r} = 6 & \frac{\partial u}{\partial s} = 5 & \frac{\partial v}{\partial r} = 4 & \frac{\partial v}{\partial t} = 3 & \frac{\partial w}{\partial s} = 2 & \frac{\partial w}{\partial t} = 1 \end{array}$$

- a. 163
- b. 212
- c. 358 **Correct Choice**
- d. 396
- e. 408

Solution

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v} \frac{\partial v}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial u} \frac{\partial u}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial w} \frac{\partial w}{\partial s} \\ &= 2 \cdot 5 \cdot 5 + 3 \cdot 8 \cdot 2 + 4 \cdot 9 \cdot 5 + 4 \cdot 10 \cdot 2 = 358 \end{aligned}$$

4. The point $(x,y) = (1,2)$ is a critical point of the function $f = 8x^3 + y^3 - 12xy$.
Use the 2nd Derivative Test to classify it as:

- a. Local Minimum **Correct Choice**
- b. Local Maximum
- c. Inflection Point
- d. Saddle Point
- e. The 2nd Derivative Test FAILS.

Solution: $f_x = 24x^2 - 12y$ $f_y = 3y^2 - 12x$ $f_x(1,2) = 24 - 12 \cdot 2 = 0$ $f_y(1,2) = 3 \cdot 4 - 12 = 0$
 $f_{xx} = 48x$ $f_{xx}(1,2) = 48$ $f_{yy} = 6y$ $f_{yy}(1,2) = 12$ $f_{xy} = -12$
 $D = f_{xx}f_{yy} - f_{xy}^2 = 48 \cdot 12 - 12^2 = 432$
 $D > 0$ and $f_{xx} > 0$. So this is a local minimum.

5. If x , y and z are related by $x \cos y + z \sin y = 3$. Find $\frac{\partial z}{\partial x}$ at the point $(x, y, z) = \left(3, \frac{\pi}{3}, \sqrt{3}\right)$.

a. $\frac{1}{\sqrt{3}}$

b. $\frac{-1}{\sqrt{3}}$ Correct Choice

c. $\sqrt{3}$

d. $-\sqrt{3}$

e. $\frac{1}{3}$

Solution: Apply $\frac{\partial}{\partial x}$: $\cos y + \frac{\partial z}{\partial x} \sin y = 0$ $\frac{1}{2} + \frac{\partial z}{\partial x} \frac{\sqrt{3}}{2} = 0$ $\frac{\partial z}{\partial x} = \frac{-1}{\sqrt{3}}$

6. If x , y and z are related by $x \cos y + z \sin y = 3$. Find $\frac{\partial z}{\partial t}$ at the instant when:

$$(x, y, z) = \left(3, \frac{\pi}{3}, \sqrt{3}\right) \quad \frac{dx}{dt} = \sqrt{3} \quad \frac{dy}{dt} = 1$$

a. 1 Correct Choice

b. 2

c. 3

d. $\sqrt{3}$

e. $\frac{1}{\sqrt{3}}$

Solution: Apply $\frac{d}{dt}$: $\frac{dx}{dt} \cos y - x \sin y \frac{dy}{dt} + \frac{\partial z}{\partial t} \sin y + z \cos y \frac{dy}{dt} = 0$

Plug in numbers: $\sqrt{3} \frac{1}{2} - 3 \frac{\sqrt{3}}{2} 1 + \frac{\partial z}{\partial t} \frac{\sqrt{3}}{2} + \sqrt{3} \frac{1}{2} 1 = 0$

Multiply by $\frac{2}{\sqrt{3}}$: $1 - 3 + \frac{\partial z}{\partial t} + 1 = 0$ $\frac{\partial z}{\partial t} = 1$

7. Find the tangent plane to the graph of the equation $xy - zy = 4$ at the point $(x, y, z) = (3, 2, 1)$. Its z -intercept is:

a. $c = -8$

b. $c = -4$ Correct Choice

c. $c = 0$

d. $c = 4$

e. $c = 8$

Solution: Let $f = xy - zy$ and $P = (3, 2, 1)$. Then $\vec{\nabla} f = \langle y, x - z, -y \rangle$

$\vec{N} = \vec{\nabla} f|_P = \langle 2, 2, -2 \rangle$ $\vec{N} \cdot X = \vec{N} \cdot P$ $2x + 2y - 2z = 2 \cdot 3 + 2 \cdot 2 - 2 \cdot 1 = 8$

The z -intercept is $c = \frac{8}{-2} = -4$.

8. Queen Lena is flying the Centurion Eagle through a deadly Sythion field whose density is $S = xyz \frac{\text{Sythions}}{\text{microlightyear}^3}$. The top speed of the Centurion Eagle is $14 \frac{\text{microlightyears}}{\text{lightyear}}$.

If Lena is located at the point $(x,y,z) = (1,2,3)$, what should her velocity be to **decrease** the Sythion density as fast as possible?

- a. $\langle 6, 3, 2 \rangle$
- b. $\langle -84, -42, -28 \rangle$
- c. $\langle -6, -3, -2 \rangle$
- d. $\langle 12, 6, 4 \rangle$
- e. $\langle -12, -6, -4 \rangle$ Correct Choice

Solution:

$$\vec{\nabla} S = \langle yz, xz, xy \rangle \quad \vec{\nabla} S \Big|_{(1,2,3)} = \langle 6, 3, 2 \rangle \quad |\vec{\nabla} S| = \sqrt{36 + 9 + 4} = 7 \quad \widehat{\nabla} S = \frac{\vec{\nabla} S}{|\vec{\nabla} S|} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle$$

The direction of maximum decrease is $\vec{v} = -\widehat{\nabla} S = \left\langle \frac{-6}{7}, \frac{-3}{7}, \frac{-2}{7} \right\rangle$. The maximum speed is $|\vec{v}| = 14$.

So the velocity of maximum decrease is $\vec{v} = |\vec{v}| \widehat{v} = 14 \left\langle \frac{-6}{7}, \frac{-3}{7}, \frac{-2}{7} \right\rangle = \langle -12, -6, -4 \rangle$.

9. Consider the limit: $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y^2}{x^3 + y^6}$. Which of the following directions of approach gives a different value of the limit?

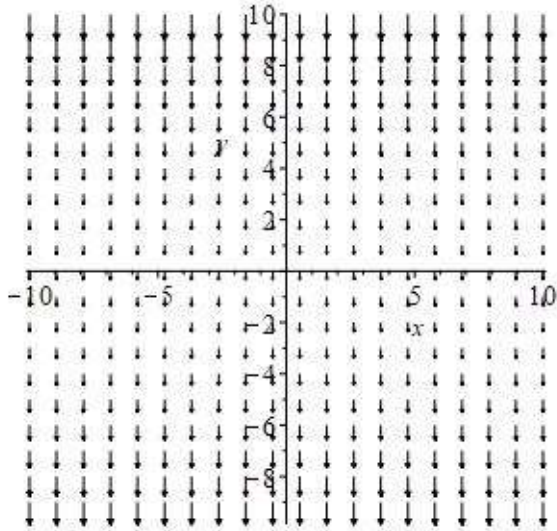
- a. Non-vertical line: $y = mx$ and $x \rightarrow 0$
- b. The y -axis: $x = 0$ and $y \rightarrow 0$
- c. The parabola: $x = y^2$ and $y \rightarrow 0$ Correct Choice
- d. The parabola: $y = x^2$ and $x \rightarrow 0$
- e. None of these. They all give the same limit.

Solution: (a) $\lim_{\substack{y=mx \\ x \rightarrow 0}} \frac{x^2 y^2}{x^3 + y^6} = \lim_{x \rightarrow 0} \frac{x^2 m^2 x^2}{x^3 + m^6 x^6} = \lim_{x \rightarrow 0} \frac{x}{1 + m^6 x^3} = 0$ (b) $\lim_{\substack{x=0 \\ y \rightarrow 0}} \frac{x^2 y^2}{x^3 + y^6} = \lim_{y \rightarrow 0} \frac{0}{0 + y^6} = 0$

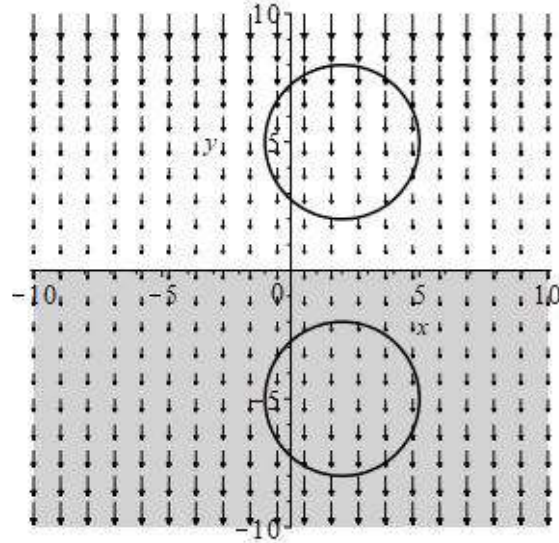
(c) $\lim_{\substack{x=y^2 \\ y \rightarrow 0}} \frac{x^2 y^2}{x^3 + y^6} = \lim_{y \rightarrow 0} \frac{y^4 y^2}{y^6 + y^6} = \lim_{y \rightarrow 0} \frac{y^6}{2y^6} = \frac{1}{2}$ (d) $\lim_{\substack{y=x^2 \\ x \rightarrow 0}} \frac{x^2 y^2}{x^3 + y^6} = \lim_{x \rightarrow 0} \frac{x^2 x^4}{x^3 + x^{12}} = \lim_{x \rightarrow 0} \frac{x^3}{1 + x^9} = 0$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (5 points) Here is the plot of a vector field \vec{F} in \mathbb{R}^2 .
Shade in the region where $\vec{\nabla} \cdot \vec{F} > 0$. Explain why.



Solution: On the bottom half of the plot larger arrows point out than in on each circle. So on the bottom, $\vec{\nabla} \cdot \vec{F} > 0$.



11. (20 points) Find a scalar potential, f , for $\vec{F} = \langle yz^2 - 2xz, xz^2 - 3y^2z, 2xyz - x^2 - y^3 + 2z \rangle$ or show one does not exist. Explain all steps neatly and clearly.

Solution: $\vec{\nabla}f = \vec{F}$ (1) $\partial_x f = yz^2 - 2xz$ (2) $\partial_y f = xz^2 - 3y^2z$ (3) $\partial_z f = 2xyz - x^2 - y^3 + 2z$

$$(1) \Rightarrow f = xyz^2 - x^2z + g(y, z)$$

$$(2) \Rightarrow \partial_y f = xz^2 - 3y^2z = xz^2 + \partial_y g \quad \partial_y g = -3y^2z \quad g = -y^3z + h(z) \quad f = xyz^2 - x^2z - y^3z + h(z)$$

$$(3) \Rightarrow \partial_z f = 2xyz - x^2 - y^3 + 2z = 2xyz - x^2 - y^3 + \frac{dh}{dz} \quad \frac{dh}{dz} = 2z \quad h = z^2$$

$$f = xyz^2 - x^2z - y^3z + z^2$$

12. (25 points) Find the largest and smallest values of the function $f(x,y,z) = xyz$ on the ellipsoid $x^2 + 4y^2 + 16z^2 = 48$.

Solution Method 1: Lagrange Multipliers:

Let $g = x^2 + 4y^2 + 16z^2$.

$$\vec{\nabla}f = \langle yz, xz, xy \rangle \quad \vec{\nabla}g = \langle 2x, 8y, 32z \rangle$$

Lagrange equations: $\vec{\nabla}f = \lambda \vec{\nabla}g$

$$yz = \lambda 2x \quad xz = \lambda 8y \quad xy = \lambda 32z$$

Multiply first equation by x . Second by y . Third by z .

$$xyz = \lambda 2x^2 \quad xyz = \lambda 8y^2 \quad xyz = \lambda 32z^2$$

Equate and divide by 2λ :

$$x^2 = 4y^2 = 16z^2$$

Substitute into the ellipsoid:

$$x^2 + x^2 + x^2 = 48 \quad x^2 = 16 \quad x = \pm 4 \quad y = \pm 2 \quad z = \pm 1$$

The function values are

$$f = xyz = (\pm 4)(\pm 2)(\pm 1) = \pm 8$$

Maximum is $f = 8$. Minimum is $f = -8$.

Solution Method 2: Eliminate a Variable:

It is easier to extremize $F = f^2 = x^2y^2z^2$ and then take a square root.

We solve the constraint for $x^2 = 48 - 4y^2 - 16z^2$ and plug into F :

$$F = (48 - 4y^2 - 16z^2)y^2z^2 = 48y^2z^2 - 4y^4z^2 - 16y^2z^4$$

$$F_y = 96yz^2 - 16y^3z^2 - 32yz^4 = 16yz^2(6 - y^2 - 2z^2) = 0$$

$$F_z = 96y^2z - 8y^4z - 64y^2z^3 = 8y^2z(12 - y^2 - 8z^2) = 0$$

Case 1: $y = 0$ Then $f = 0$. Note: x and z are anything satisfying $x^2 + 16z^2 = 48$.

Case 2: $z = 0$ Then $f = 0$. Note: x and y are anything satisfying $x^2 + 4y^2 = 48$.

Case 3: $y \neq 0$ and $z \neq 0$ Then

$$y^2 + 2z^2 = 6 \quad (1)$$

$$y^2 + 8z^2 = 12 \quad (2)$$

(2) - (1) says: $6z^2 = 6$ or $z = \pm 1$.

Then (1) says: $y^2 = 4$ or $y = \pm 2$.

Then the constraint says: $x^2 = 48 - 4y^2 - 16z^2 = 48 - 16 - 16 = 16$ or $x = \pm 4$

The function values are

$$f = xyz = (\pm 4)(\pm 2)(\pm 1) = \pm 8$$

Maximum is $f = 8$. Minimum is $f = -8$.

The $f = 0$ values from Cases 1 and 2 don't matter.