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MATH 221 Exam 3 Version A Fall 2019

Section 504 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-9	/54	11	/25
10	/25	Total	/104

1. Compute the integral $\iint x^2 y dA$ over the region between $y = 4x^2$ and $y = x^4$ in the first quadrant.

- a. $\frac{2^{12}}{77}$ Correct Choice
- b. $\frac{2^{13}}{77}$
- c. $\frac{2^6}{35}$
- d. $\frac{2^7}{35}$
- e. $\frac{2^8}{35}$

Solution: The curves intersect when $4x^2 = x^4$ or $x = 0, \pm 2$. At $x = 1$, $4x^2 > x^4$. So

$$\begin{aligned} \iint x^2 y dA &= \int_0^2 \int_{x^4}^{4x^2} x^2 y dy dx = \int_0^2 \left[x^2 \frac{y^2}{2} \right]_{y=x^4}^{4x^2} dx = \int_0^2 \left(x^2 \frac{16x^4}{2} \right) - \left(x^2 \frac{x^8}{2} \right) dx \\ &= \left[8 \frac{x^7}{7} - \frac{1}{2} \frac{x^{11}}{11} \right]_0^2 = \frac{2^{10}}{7} - \frac{2^{10}}{11} = \frac{2^{12}}{77} \end{aligned}$$

2. Find the average value of the function $f(x,y) = x^2 y$ on the rectangle $[0,3] \times [0,4]$.

- a. $\frac{3}{2}$
- b. 6 Correct Choice
- c. 12
- d. 18
- e. 72

Solution: The area is $A = 3 \cdot 4 = 12$.

$$\int_0^4 \int_0^3 x^2 y dx dy = \left[\frac{x^3}{3} \right]_0^3 \left[\frac{y^2}{2} \right]_0^4 = 9 \cdot 8 = 72 \quad f_{\text{ave}} = \frac{1}{A} \int_0^4 \int_0^3 x^2 y dx dy = \frac{72}{12} = 6$$

3. Compute $\int_0^4 \int_{\sqrt{y}}^2 e^{x^3} dx dy$

a. $\frac{e^8}{3} - \frac{1}{3}$ Correct Choice

b. $\frac{e^{16}}{4} - \frac{1}{4}$

c. $\frac{e^{16}}{4} - \frac{e^4}{4}$

d. $\frac{e^{64}}{4} - \frac{1}{4}$

e. $\frac{e^{64}}{4} - \frac{e^{16}}{4}$

Solution: We reverse the order of integration.

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3} dx dy = \int_0^2 \int_0^{x^2} e^{x^3} dy dx = \int_0^2 x^2 e^{x^3} dx = \left[\frac{1}{3} e^{x^3} \right]_0^2 = \frac{e^8}{3} - \frac{1}{3}$$

4. Find the area of the **inner loop** of the limaçon $r = 4 \cos \theta - 2$.

HINT: Find the angles at which $r = 0$.

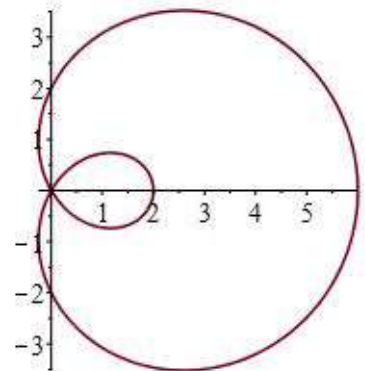
a. $2\pi + 6\sqrt{3}$

b. $4\pi - 6\sqrt{3}$ Correct Choice

c. $6\pi - 4\sqrt{3}$

d. $8\pi + 4\sqrt{3}$

e. $8\pi + 6\sqrt{3}$



Solution: The loop passes thru the origin when $r = 0$. So

$$4 \cos \theta - 2 = 0 \quad \cos \theta = \frac{1}{2} \quad \theta = \pm \frac{\pi}{3}$$

$$\begin{aligned} A &= \int_{-\pi/3}^{\pi/3} \int_0^{4 \cos \theta - 2} r dr d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (4 \cos \theta - 2)^2 d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (16 \cos^2 \theta - 16 \cos \theta + 4) d\theta \\ &= \frac{1}{2} \int_{-\pi/3}^{\pi/3} (16 \cos^2 \theta - 16 \cos \theta + 4) d\theta = \frac{1}{2} \int_{-\pi/3}^{\pi/3} (8(1 + \cos 2\theta) - 16 \cos \theta + 4) d\theta \\ &= \frac{1}{2} [12\theta + 4 \sin 2\theta - 16 \sin \theta]_{-\pi/3}^{\pi/3} = 12 \frac{\pi}{3} + 4 \sin \frac{2\pi}{3} - 16 \sin \frac{\pi}{3} = 4\pi - 6\sqrt{3} \end{aligned}$$

5. Find the mass of the solid between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$ if the density is $\delta = z$.
- 4π
 - 32π
 - 36π
 - 64π Correct Choice
 - 120π

Solution: In cylindrical coordinates, the paraboloids are $z = r^2$ and $z = 8 - r^2$. They intersect when

$$r^2 = 8 - r^2 \quad r^2 = 4 \quad r = 2$$

$$\begin{aligned} M &= \iiint \delta dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} zr dz dr d\theta = 2\pi \int_0^2 \left[\frac{z^2}{2} \right]_{z=r^2}^{8-r^2} r dr = \pi \int_0^2 [(8-r^2)^2 - r^4] r dr \\ &= \pi \int_0^2 [64 - 16r^2] r dr = \pi [32r^2 - 4r^4]_0^2 = \pi(128 - 64) = 64\pi \end{aligned}$$

6. Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ below the sphere $x^2 + y^2 + z^2 = 9$.
- 9π
 - 18π
 - $9\pi\sqrt{2}$
 - $9\pi\left(1 - \frac{\sqrt{3}}{2}\right)$
 - $18\pi\left(1 - \frac{1}{\sqrt{2}}\right)$ Correct Choice

Solution: In spherical coordinates, the sphere is $\rho = 3$ and the cone is $\rho \cos \varphi = \rho \sin \varphi$ or $\varphi = \frac{\pi}{4}$.

$$V = \iiint 1 dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^3 \rho^2 \sin \varphi d\rho d\varphi d\theta = 2\pi \left[\frac{\rho^3}{3} \right]_0^3 \left[-\cos \varphi \right]_0^{\pi/4} = 18\pi \left(1 - \frac{1}{\sqrt{2}} \right)$$

7. Compute the circulation of the vector field $\vec{F} = \langle -yz, xz, z^2 \rangle$ counterclockwise around the circle $\vec{r}(\theta) = (3 \cos \theta, 3 \sin \theta, 2)$. Note: $Circ = \oint \vec{F} \cdot d\vec{s}$
- 4π
 - 9π
 - 36π Correct Choice
 - 54π
 - 72π

Solution: $\vec{F}(\vec{r}(\theta)) = \langle -6 \sin \theta, 6 \cos \theta, 4 \rangle$ $\vec{v} = \langle -3 \sin \theta, 3 \cos \theta, 0 \rangle$

$$Circ = \oint \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v} d\theta = \int_0^{2\pi} (18 \sin^2 \theta + 18 \cos^2 \theta) d\theta = 36\pi$$

8. Find the x -component of the center of mass of the twisted helix $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ for $0 \leq t \leq 3$, if the density is $\delta = 2$.
- $\bar{x} = 42$
 - $\bar{x} = 90$
 - $\bar{x} = \frac{15}{14}$
 - $\bar{x} = \frac{14}{15}$
 - $\bar{x} = \frac{15}{7}$ Correct Choice

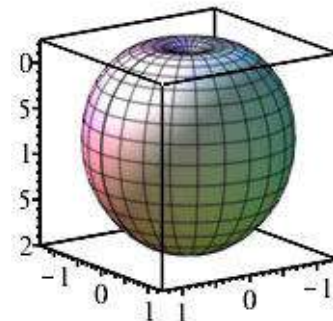
Solution: $\vec{v} = (1, 2t, 2t^2)$ $|\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = 1 + 2t^2$
 $M = \int \delta |\vec{v}| dt = \int_0^3 2(1 + 2t^2) dt = \left[2t + \frac{4}{3}t^3\right]_0^3 = 6 + 36 = 42$
 $M_x = \int x \delta |\vec{v}| dt = \int_0^3 t \cdot 2(1 + 2t^2) dt = [t^2 + t^4]_0^3 = 9 + 81 = 90$
 $\bar{x} = \frac{M_x}{M} = \frac{90}{42} = \frac{15}{7}$

9. Compute $\iiint \vec{\nabla} \cdot \vec{F} dV$ over the solid between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 16$ if $\vec{F} = \langle xz^2, yz^2, z^3 \rangle$.
- $\frac{4^6\pi}{3}$
 - $\frac{2^7\pi}{3}$
 - $\frac{4\pi}{3}(4^5 - 2^5)$ Correct Choice
 - $\frac{2\pi}{3}(4^5 - 2^5)$
 - $\frac{2\pi}{3}(4^6 - 2^6)$

Solution: $\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + 3z^2 = 5z^2 = 5\rho^2 \cos^2\phi$
 $\iiint \vec{\nabla} \cdot \vec{F} dV = \int_0^{2\pi} \int_0^\pi \int_2^4 5\rho^2 \cos^2\phi \rho^2 \sin\phi d\rho d\phi d\theta$ Let $u = \cos\phi$ and $du = -\sin\phi d\phi$.
 $\iiint \vec{\nabla} \cdot \vec{F} dV = -2\pi \left[\frac{5\rho^5}{5}\right]_2^4 \int_1^{-1} u^2 du = -2\pi(4^5 - 2^5) \left[\frac{u^3}{3}\right]_1^{-1} = \frac{4\pi}{3}(4^5 - 2^5)$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (25 points) The apple at the right is given in spherical coordinates by $\rho = 1 - \cos \varphi$.



HINTS: $(1 - u)^3 = 1 - 3u + 3u^2 - u^3$
 $(1 - u)^4 = 1 - 4u + 6u^2 - 4u^3 + u^4$

- a. Find the volume.

Solution:

$$V = \iiint 1 dV = \int_0^{2\pi} \int_0^\pi \int_0^{1-\cos\varphi} \rho^2 \sin\varphi d\rho d\varphi d\theta = 2\pi \int_0^\pi \sin\varphi \left[\frac{\rho^3}{3} \right]_0^{1-\cos\varphi} d\varphi = \frac{2\pi}{3} \int_0^\pi (1 - \cos\varphi)^3 \sin\varphi d\varphi$$

Let $u = \cos\varphi$. So $du = -\sin\varphi d\varphi$. Then

$$V = \frac{-2\pi}{3} \int_1^{-1} (1 - u)^3 du = \frac{2\pi}{3} \int_{-1}^1 (1 - 3u + 3u^2 - u^3) du = \frac{2\pi}{3} \left[u - \frac{3u^2}{2} + u^3 - \frac{u^4}{4} \right]_{-1}^1$$

Even powers cancel. Odd powers add up.

$$V = \frac{4\pi}{3} (1 + 1^3) = \frac{8\pi}{3}$$

- b. Find the centroid. (Part credit: Set up the integrals and say what you do with the answers.)

Solution: $\bar{x} = \bar{y} = 0$ by symmetry.

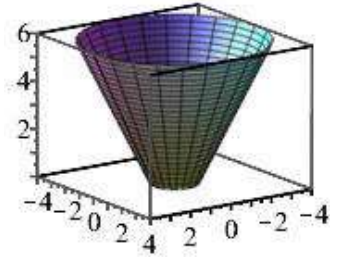
$$V_z = \iiint z dV = \int_0^{2\pi} \int_0^\pi \int_0^{1-\cos\varphi} \rho \cos\varphi \rho^2 \sin\varphi d\rho d\varphi d\theta = 2\pi \int_0^\pi \cos\varphi \sin\varphi \left[\frac{\rho^4}{4} \right]_0^{1-\cos\varphi} d\varphi$$

$$= \frac{\pi}{2} \int_0^\pi \cos\varphi (1 - \cos\varphi)^4 \sin\varphi d\varphi = \frac{-\pi}{2} \int_1^{-1} u(1 - u)^4 du = \frac{\pi}{2} \int_{-1}^1 u(1 - 4u + 6u^2 - 4u^3 + u^4) du$$

$$= \frac{\pi}{2} \left[\frac{u^2}{2} - \frac{4u^3}{3} + \frac{6u^4}{4} - \frac{4u^5}{5} + \frac{u^6}{6} \right]_{-1}^1 = \pi \left(-\frac{4}{3} - \frac{4}{5} \right) = -\frac{32}{15} \pi$$

$$\bar{z} = \frac{V_z}{V} = -\frac{32\pi}{15} \frac{3}{8\pi} = -\frac{4}{5}$$

11. (25 points) Consider a bowl given in cylindrical coordinates by $z = 2r - 2$ for $1 \leq r \leq 4$ oriented **down and out**. The surface may be parametrized by $\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, 2r - 2)$.



a. Find the surface area.

i. Find the tangent vectors:

$$\vec{e}_r = (\cos \theta, \sin \theta, 2)$$

$$\vec{e}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

ii. Find the normal:

$$\vec{N} = \hat{i}(-2r \cos \theta) - \hat{j}(2r \sin \theta) + \hat{k}(r)$$

iii. Find the length of the normal:

$$|\vec{N}| = \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + r^2} = \sqrt{4r^2 + r^2} = \sqrt{5} r$$

iv. Find the area:

$$A = \iint 1 dS = \iint 1 |\vec{N}| dr d\theta = \int_0^{2\pi} \int_1^4 \sqrt{5} r dr d\theta = 2\pi \sqrt{5} \left[\frac{r^2}{2} \right]_1^4 = \pi \sqrt{5} (16 - 1) = 15\pi \sqrt{5}$$

b. Find the flux of the vector field $\vec{F} = \langle x, y, -z \rangle$ **down** through the surface.

i. Evaluate the vector field on the surface:

$$\vec{F}|_{\vec{R}} = \langle r \cos \theta, r \sin \theta, -2r + 2 \rangle$$

ii. Restate the normal:

$$\vec{N} = (-2r \cos \theta, -2r \sin \theta, r) \quad \text{This is up and in.}$$

$$\text{Reverse: } \vec{N} = (2r \cos \theta, 2r \sin \theta, -r) \quad \text{This is down and out.}$$

iii. Compute the flux:

$$\begin{aligned} \text{Flux} &= \iint \vec{F} \cdot d\vec{S} = \iint \vec{F}|_{\vec{R}} \cdot \vec{N} dr d\theta = \int_0^{2\pi} \int_1^4 (2r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + (2r - 2)r) dr d\theta \\ &= \int_0^{2\pi} \int_1^4 (4r^2 - 2r) dr d\theta = 2\pi \left[\frac{4r^3}{3} - r^2 \right]_1^4 = 2\pi \left(\frac{256}{3} - 16 \right) - 2\pi \left(\frac{4}{3} - 1 \right) \\ &= 2\pi \left(\frac{252}{3} - 15 \right) = 2\pi(84 - 15) = 138\pi \end{aligned}$$