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1-10	/50	12	/10
11	/10	13	/35
		Total	/105

MATH 221 Exam 1 Fall 2021

Sections 504/505 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. A point is given in cylindrical coordinates by $(r,\theta,z) = \left(3, \frac{\pi}{3}, 3\right)$.
Find its spherical coordinates.

$$(\rho,\varphi,\theta) = (\_\_\_\_,\_\_\_\_,\_\_\_\_)$$

Solution: $\rho = \sqrt{r^2+z^2} = \sqrt{3^2+3^2} = \sqrt{18} = 3\sqrt{2}$ $\qquad \cos\varphi = \frac{z}{\rho} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$

$\varphi = \arccos\frac{1}{\sqrt{2}} = \frac{\pi}{4}$

$$(\rho,\varphi,\theta) = \left(\_\_3\sqrt{2}\_\_,\_\_\frac{\pi}{4}\_\_,\_\_\frac{\pi}{3}\_\_\right)$$

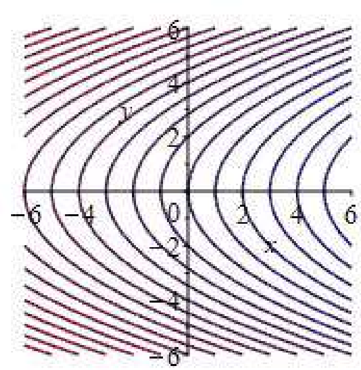
2. A sphere is centered at $(1,3,5)$ and is tangent to the plane $y = 1$.
What is the equation of the sphere?

a. $(x-1)^2+(y-3)^2+(z-5)^2 = 0$
b. $(x-1)^2+(y-3)^2+(z-5)^2 = 1$
c. $(x-1)^2+(y-3)^2+(z-5)^2 = 2$
d. $(x-1)^2+(y-3)^2+(z-5)^2 = 3$
e. $(x-1)^2+(y-3)^2+(z-5)^2 = 4$ Correct
f. $(x-1)^2+(y-3)^2+(z-5)^2 = 5$
g. $(x-1)^2+(y-3)^2+(z-5)^2 = 6$
h. $(x-1)^2+(y-3)^2+(z-5)^2 = 7$
i. $(x-1)^2+(y-3)^2+(z-5)^2 = 8$
j. $(x-1)^2+(y-3)^2+(z-5)^2 = 9$

Solution: The radius is the distance from the center $(1,3,5)$ to the plane $y = 1$ which is $r = 3-1 = 2$. So the equation is

$$(x-a)^2+(y-b)^2+(z-c)^2 = r^2$$
$$(x-1)^2+(y-3)^2+(z-5)^2 = 4$$

3. This is the contour plot of which function?



a. $f(x,y) = y - \frac{x^2}{4}$

b. $f(x,y) = y + \frac{x^2}{4}$

c. $f(x,y) = x - \frac{y^2}{4}$ Correct Choice

d. $f(x,y) = x + \frac{y^2}{4}$

Solution: Each contour is a curve $f(x,y) = C$.

$y - \frac{x^2}{4} = C$ is a parabola opening up. $\qquad y - \frac{x^2}{4} = C$ is a parabola opening down.

$x - \frac{y^2}{4} = C$ is a parabola opening right. $\qquad x - \frac{y^2}{4} = C$ is a parabola opening left.

4. Write $\langle 5,5,5\rangle$ as a linear combination of $\langle 3,-1,2\rangle$ and $\langle 1,3,2\rangle$ or type "impossible" in both boxes.

$$\langle 5,5,5\rangle = \_\_\_\_\langle 3,-1,2\rangle + \_\_\_\_\langle 1,3,2\rangle$$

Solution: Let $\langle 5,5,5\rangle = a\langle 3,-1,2\rangle + b\langle 1,3,2\rangle$. Then

$$(1) \qquad 5 = 3a + b$$
$$(2) \qquad 5 = -a + 3b$$
$$(3) \qquad 5 = 2a + 2b$$

From (1): $b = 5 - 3a$ Then (2) says: $5 = -a + 3(5-3a) = -10a + 15$ or $10a = 10$ or $a = 1$ and so $b = 5 - 3(1) = 2$. So (3) says $5 = 2(1) + 2(2) = 6$. Impossible!

5. Find the angle between the vectors $\langle 2,-2,1\rangle$ and $\langle 1,-4,-1\rangle$.

a. $0°$
b. $30°$
c. $45°$ Correct Choice
d. $60°$
e. $90°$
f. $120°$
g. $135°$
h. $150°$
i. $180°$

Solution: Let $\vec{a} = \langle 2,-2,1\rangle$ and $\vec{b} = \langle 1,-4,-1\rangle$. Then

$$\cos\theta = \frac{\vec{a}\bullet\vec{b}}{|\vec{a}||\vec{b}|} = \frac{2+8-1}{\sqrt{4+4+1}\sqrt{1+16=1}} = \frac{9}{3\sqrt{18}} = \frac{1}{\sqrt{2}}$$

So $\theta = 45°$

6. Write $\vec{v} = \langle 2,-8,-2\rangle$ as the sum of two vectors $\vec{p}$ and $\vec{q}$ where $\vec{p}$ is parallel to $\vec{u} = \langle 2,-2,1\rangle$ and $\vec{q}$ is perpendicular to $\vec{u}$.

$$\vec{v} = \langle 2,-8,-2\rangle = \vec{p} + \vec{q}$$

where

$$\vec{p} = \langle \_\_\_\_,\_\_\_\_,\_\_\_\_\rangle \quad \text{and} \quad \vec{q} = \langle \_\_\_\_,\_\_\_\_,\_\_\_\_\rangle$$

Solution: $\vec{p} = \text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\bullet\vec{v}}{|\vec{u}|^2}\vec{u} = \frac{4+16-2}{4+4+1}\langle 2,-2,1\rangle = 2\langle 2,-2,1\rangle = \langle 4,-4,2\rangle$

$\vec{q} = \text{proj}_{\perp\vec{u}}\vec{v} = \vec{v} - \text{proj}_{\vec{u}}\vec{v} = \langle 2,-8,-2\rangle - \langle 4,-4,2\rangle = \langle -2,-4,-4\rangle$

Check: $\vec{p} + \vec{q} = \langle 4,-4,2\rangle + \langle -2,-4,-4\rangle = \langle 2,-8,-2\rangle$
$\vec{p} = \langle 4,-4,2\rangle$ is a multiple of $\vec{u} = \langle 2,-2,1\rangle$ $\quad \vec{q}\bullet\vec{u} = \langle -2,-4,-4\rangle \bullet \langle 2,-2,1\rangle = -4 + 8 - 4 = 0$

7. If $\vec{a}$ points DOWN and $\vec{b}$ points SOUTHWEST, in what direction does $\vec{a} \times \vec{b}$ point?

a. NORTH
b. SOUTH
c. EAST
d. WEST
e. DOWN
f. NORTHEAST
g. NORTHWEST Correct Choice
h. SOUTHEAST
i. SOUTHWEST
j. UP

Solution: Hold the fingers of your right hand pointing DOWN with the palm facing SOUTHWEST. Then your thumb points NORTHWEST.

8. Find the volume of the parallelepiped with edge vectors

$$\vec{p} = \langle 2,1,3 \rangle \qquad \vec{q} = \langle 3,2,0 \rangle \qquad \vec{r} = \langle 4,0,1 \rangle$$

$$V = \_\_\_\_\_$$

Solution: $\vec{p} \times \vec{q} \cdot \vec{r} = \begin{vmatrix} 2 & 1 & 3 \\ 3 & 2 & 0 \\ 4 & 0 & 1 \end{vmatrix} = 4\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 4(0-6) + 1(4-3) = -24 + 1 = -23$

$V = |\vec{p} \times \vec{q} \cdot \vec{r}| = 23$

9. Find the standard equation of the plane which passes through the point $P = (3,2,1)$ and is perpendicular to the line $\vec{r}(t) = (2+t, 3-2t, 1+4t)$.

$$\_\_\_\_x + \_\_\_\_y + \_\_\_\_z = \_\_\_\_$$

Solution: The normal to the plane is the direction of the line which is the coefficients of t in its equation: $\vec{N} = \vec{v} = \langle 1,-2,4 \rangle$ Then the equation of the plane is:

$$\vec{N} \cdot X = \vec{N} \cdot P$$

$$1x - 2y + 4z = 1(3) - 2(2) + 4(1) = 3$$

10. Identify the surface

$$9x^2 - 36x - 4y^2 + 8y + z^2 + 4z + 36 = 0$$

a. Sphere
b. Ellipsoid
c. Hyperboloid of 1 sheet
d. Hyperboloid of 2 sheets
e. Cone Correct Choice
f. Elliptic Paraboloid
g. Hyperbolic Paraboloid
h. Elliptic Cylinder
i. Hyperbolic Cylinder
j. Parabolic Cylinder

Solution: The plusses in front of x^2 and z^2 and the minus in front of y^2 say it is a hyperboloid or cone. We complete the squares:

$$9x^2 - 36x - 4y^2 + 8y + z^2 + 4z + 36 = 0$$
$$9(x^2 - 4x) - 4(y^2 - 2y) + (z^2 + 4z) = -36$$
$$9(x^2 - 4x + 4) - 4(y^2 - 2y + 1) + (z^2 + 4z + 4) = -36 + 36 - 4 + 4$$
$$9(x-2)^2 - 4(y-1)^2 + (z+2)^2 = 0$$

So it is a cone.

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (10 points) Find the point of intersection of the line

$$\frac{x+2}{2} = \frac{y+5}{3} = \frac{z+5}{4}$$

and the plane:

$$x - y + z = 4$$

(You will be graded on your work.)

Solution: We convert the line into parametric form:

$$\frac{x+2}{2} = \frac{y+5}{3} = \frac{z+5}{4} = t$$

$$x = -2 + 2t \qquad y = -5 + 3t \qquad z = -5 + 4t$$

Then we plug into the plane to find the value of t:

$$x - y + z = 4$$
$$(-2 + 2t) - (-5 + 3t) + (-5 + 4t) = 4$$
$$-2 + 3t = 4$$
$$t = 2$$

Finally, we plug $t = 2$ into the parametric line:

$$x = -2 + 2(2) = 2 \qquad y = -5 + 3(2) = 1 \qquad z = -5 + 4(2) = 3$$

So the point is:

$$P = (2, 1, 3)$$

To check we plug the point into the plane:

$$x - y + z = 2 - 1 + 3 = 4$$

12. (10 points) Consider the two planes

$$P_1 : \quad x + y + z = 3$$
$$P_2 : \quad x - y + 2z = 1$$

Compute each of the following quantities. (You will be graded on your work.)

a. The normal vectors to the planes:

$\vec{N}_1 =$ ____________________

$\vec{N}_2 =$ ____________________

Solution: Read off coefficients of x, y and z:
$\vec{N}_1 = \langle 1,1,1 \rangle$
$\vec{N}_2 = \langle 1,-1,2 \rangle$

b. The direction of the line of intersection:

$\vec{v} =$ ____________________

Solution: $\vec{v} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \hat{\imath}(2 - -1) - \hat{\jmath}(2 - 1) + \hat{k}(-1 - 1) = \langle 3,-1,-2 \rangle$

c. A point on the line of intersection:

$P =$ ____________________

Solution: Set $z = 0$ in both planes and solve: $x + y = 3 \quad x - y = 1$
So $2x = 4 \quad x = 2 \quad y = 1.$ So $P = (2,1,0)$

d. The equation of the line of intersection:

$\vec{r}(t) =$ ____________________

Solution: $\vec{r}(t) = P + t\vec{v} = (2,1,0) + t\langle 3,-1,-2 \rangle = (2 + 3t, 1 - t, -2t)$

13. (35 points) Consider the parametric curve $\vec{r}(t) = \left(t^2, \frac{2}{3}t^3, \frac{1}{4}t^4\right)$.
Compute each of the following quantities. (You will be graded on your work.)

a. Velocity $\vec{v} =$

Solution: $\vec{v} = \langle 2t, 2t^2, t^3 \rangle$

b. Acceleration $\vec{a} =$

Solution: $\vec{a} = \langle 2, 4t, 3t^2 \rangle$

c. Speed $|\vec{v}| =$

Solution: $|\vec{v}| = \sqrt{4t^2 + 4t^4 + t^6} = t\sqrt{4 + 4t^2 + t^4} = t\sqrt{(2 + t^2)^2} = t(2 + t^2) = 2t + t^3$

d. Unit Tangent Vector $\hat{T} =$

Solution: $\hat{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{t(2 + t^2)}\langle 2t, 2t^2, t^3 \rangle = \frac{1}{2 + t^2}\langle 2, 2t, t^2 \rangle = \left\langle \frac{2}{2 + t^2}, \frac{2t}{2 + t^2}, \frac{t^2}{2 + t^2} \right\rangle$

e. Arc Length between $A = (0,0,0)$ and $B = \left(4, \frac{16}{3}, 4\right)$ $L =$

Solution: $L = \int_A^B ds = \int_0^2 |\vec{v}|\,dt = \int_0^2 (2t + t^3)\,dt = \left[t^2 + \frac{t^4}{4}\right]_0^2 = 4 + 4 = 8$

f. The Scalar Line Integral of $f(x,y,z) = x$ between $A = (0,0,0)$ and $B = \left(4, \frac{16}{3}, 4\right)$
$\int_A^B f\,ds =$

Solution: $f(\vec{r}(t)) = x(t) = t^2$
$\int_A^B f\,ds = \int_0^2 f(\vec{r}(t))\,|\vec{v}|\,dt = \int_0^2 f(\vec{r}(t))\,|\vec{v}|\,dt = \int_0^2 t^2\,t(2 + t^2)\,dt = \int_0^2 (2t^3 + t^5)\,dt = \left[\frac{t^4}{2} + \frac{t^6}{6}\right]_0^2$
$= 8 + \frac{32}{3} = \frac{56}{3}$

g. The Vector Line Integral of $\vec{F}(x,y,z) = \langle 4z, 3y, 2x \rangle$ between $A = (0,0,0)$ and $B = \left(4, \frac{16}{3}, 4\right)$
$\int_A^B \vec{F} \cdot d\vec{s} =$

Solution: Recall: $x = t^2$, $y = \frac{2}{3}t^3$, $z = \frac{1}{4}t^4$ and $\vec{v} = \langle 2t, 2t^2, t^3 \rangle$.
So $\vec{F}(\vec{r}(t)) = \langle t^4, 2t^3, 2t^2 \rangle$ and $\vec{F} \cdot \vec{v} = 2t^5 + 4t^5 + 2t^5 = 8t^5$. So
$\int_A^B \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{v}\,dt = \int_0^2 8t^5\,dt = \left[\frac{4t^6}{3}\right]_0^2 = \frac{256}{3}$