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1-9	/45	11	/20
10	/20	12	/20
		Total	/105

MATH 221 Exam 3 Fall 2021

Sections 504/505 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $I = \int_0^4 \int_0^3 \int_0^2 x^3 y^2 z \, dz \, dy \, dx$. Simplify to an integer.

$I =$ ______________

Solution: $\int_0^4 \int_0^3 \int_0^2 x^3 y^2 z \, dz \, dy \, dx = \int_0^4 x^3 \, dx \int_0^3 y^2 \, dy \int_0^2 z \, dz = \left[\frac{x^4}{4} \right]_0^4 \left[\frac{y^3}{3} \right]_0^3 \left[\frac{z^2}{2} \right]_0^2$

$= (64)(9)(2) = \underline{1152}$

2. Find the mass of the triangle with vertices $(0,-3)$, $(0,3)$ and $(3,0)$ if the density is $\delta = x$. Simplify to an integer.

$M =$ ______________

Solution: The upper edge is $y = 3 - x$. The lower edge is $y = -3 + x$.

$$M = \iint \delta \, dA = \int_0^3 \int_{-3+x}^{3-x} x \, dy \, dx = \int_0^3 x \Big[y \Big]_{y=-3+x}^{3-x} dx = \int_0^3 x[(3-x) - (-3+x)] \, dx = 2 \int_0^3 x(3-x) \, dx$$

$$= 2 \int_0^3 (3x - x^2) \, dx = 2 \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_{x=0}^3 = 2\left(\frac{27}{2} - 9 \right) = \underline{9}$$

3. Find the x-component of the center of mass of the triangle with vertices $(0,-3)$, $(0,3)$ and $(3,0)$, if the density is $\delta = x$.
Simplify to a rational number. Enter $\frac{7}{5}$ as 7/5.

$\bar{x} =$ ______________

Solution: The mass was found in problem 2. The x-moment is

$$M_x = \iint x\delta \, dA = \int_0^3 \int_{-3+x}^{3-x} x^2 \, dy \, dx = \int_0^3 x^2 \Big[y \Big]_{y=-3+x}^{3-x} dx = \int_0^3 x^2[(3-x) - (-3+x)] \, dx = 2 \int_0^3 x^2(3-x) \, dx$$

$$= 2 \int_0^3 (3x^2 - x^3) \, dx = 2 \left[x^3 - \frac{x^4}{4} \right]_{x=0}^3 = 2\left(27 - \frac{81}{4} \right) = \frac{27}{2} \qquad \bar{x} = \frac{M_x}{M} = \frac{27}{2 \cdot 9} = \underline{\frac{3}{2}}$$

4. Estimate the double integral $I = \iint_R x^2 y\, dA$ over the rectangle $[0,4] \times [0,8]$ using a Riemann sum with 4 small rectangles which are 2 wide and 4 high with evaluation points at the center of each small rectangle.

$I \approx$ ______________

Solution: The function is $f(x,y) = x^2 y$. Each small rectangle has area $\Delta A = 2 \times 4 = 8$. The centers are $(1,2)$, $(1,6)$, $(3,2)$ and $(3,6)$. The function values are $f(1,2) = 2$, $f(1,6) = 6$, $f(3,2) = 18$ and $f(3,6) = 54$. So the Riemann sum approximation is

$$\iint_R x^2 y\, dA \approx \sum_{i=1}^{4} f(x_i^*, y_i^*)\Delta A = (2 + 6 + 18 + 54)8 = \underline{640}$$

5. Compute the integral $\int_0^1 \int_{\sqrt{y}}^1 x(x^4+1)^{24}\, dx\, dy$.

HINT: Reverse the order of integration.

a. $\frac{1}{24}(2^{23}-1)$ **e.** $\frac{1}{6}(2^{23}-1)$ **i.** $\frac{1}{96}(2^{23}-1)$

b. $\frac{1}{25}(2^{25}-1)$ **f.** $\frac{4}{25}(2^{25}-1)$ **j.** $\frac{1}{100}(2^{25}-1)$ Correct Choice

c. $\frac{1}{96}(2^{95}-1)$ **g.** $\frac{1}{24}(2^{95}-1)$ **k.** $\frac{1}{384}(2^{95}-1)$

d. $\frac{1}{97}(2^{97}-1)$ **h.** $\frac{4}{97}(2^{97}-1)$ **l.** $\frac{1}{388}(2^{97}-1)$

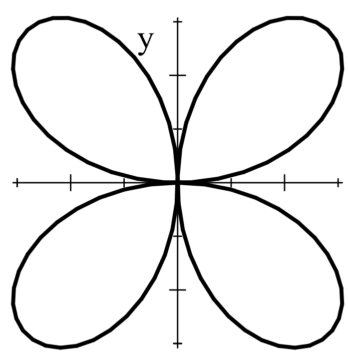
Solution: The original boundaries are $0 \le y \le 1$ and $\sqrt{y} \le x \le 1$. Draw a figure.
The new boundaries are $0 \le x \le 1$ and $0 \le y \le x^2$. The y-integral is now easy.

$$I = \int_0^1 \int_{\sqrt{y}}^1 x(x^4+1)^{24}\, dx\, dy = \int_0^1 \int_0^{x^2} x(x^4+1)^{24}\, dy\, dx = \int_0^1 x(x^4+1)^{24}\Big[y\Big]_{y=0}^{x^2} dx = \int_0^1 x^3(x^4+1)^{24}\, dx$$

We make the substitution $u = x^4 + 1$. Then $du = 4x^3\, dx$ and so $\frac{1}{4}\, du = x^3\, dx$.

$$I = \frac{1}{4}\int_1^2 u^{24}\, du = \left[\frac{1}{100}u^{25}\right]_1^2 = \frac{1}{100}(2^{25}-1)$$

6. The graph of $r = \sin(2\theta)$ is the 4-leaf clover.
Find the area of the leaf in the first quadrant.
Enter $\frac{5\pi}{6}$ as 5pi/6.



$A =$ ______________

Solution: $A = \iint 1\, dA = \int_0^{\pi/2}\int_0^{\sin(2\theta)} r\, dr\, d\theta = \int_0^{\pi/2}\left[\frac{r^2}{2}\right]_0^{\sin(2\theta)} d\theta = \frac{1}{2}\int_0^{\pi/2} \sin^2(2\theta)\, d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1-\cos(4\theta)}{2}\, d\theta = \frac{1}{4}\left[\theta - \frac{\sin(4\theta)}{4}\right]_0^{\pi/2} = \frac{1}{4}\left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) = \underline{\frac{\pi}{8}}$$

7. Find the average value of the function $f(x,y,z) = z$ over the solid P below the paraboloid $z = 9 - x^2 - y^2$ and above the xy-plane. Simplify completely.
HINT: Don't use rectangular coordinates.

$f_{\text{ave}} =$ ______________

Solution: We use cylindrical coordinates. The volume is

$$V = \iiint_P 1\,dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^3 r\Big[z\Big]_{z=0}^{9-r^2} dr = 2\pi\int_0^3 r(9-r^2)\,dr$$

$$= 2\pi\left[9\frac{r^2}{2} - \frac{r^4}{4}\right]_0^3 = 2\pi\left(\frac{81}{2} - \frac{81}{4}\right) = \frac{81\pi}{2}$$

The integral of the function is

$$\iiint_P f\,dV = \int_0^{2\pi}\int_0^3\int_0^{9-r^2} zr\,dz\,dr\,d\theta = 2\pi\int_0^3 r\left[\frac{z^2}{2}\right]_{z=0}^{9-r^2} dr = \pi\int_0^3 r(9-r^2)^2\,dr$$

Let $u = 9 - r^2$. Then $du = -2r\,dr$ and $\frac{-1}{2}du = r\,dr$. So

$$\iiint_P f\,dV = -\frac{\pi}{2}\int_9^0 u^2\,du = -\frac{\pi}{2}\left[\frac{u^3}{3}\right]_9^0 = \frac{\pi}{2}\left(\frac{9^3}{3}\right) = \frac{243\pi}{2}$$

So the average is $f_{\text{ave}} = \frac{1}{V}\iiint_P f\,dV = \frac{2}{81\pi}\frac{243\pi}{2} =$ $\underline{\;3\;}$

8. Find the z-component of the centroid of the $\frac{1}{8}$ of a sphere of radius 4 centered at the origin in the first octant (i.e. $x \ge 0$, $y \ge 0$ and $z \ge 0$).

$\bar{z} =$ ______________

Solution: We set up the volume integral:

$$V = \iiint_S 1\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho^2\sin\varphi\,d\rho\,d\phi\,d\theta$$

We don't need to compute this integral since we know the volume is $V = \frac{1}{8}\left(\frac{4}{3}\pi 4^3\right) = \frac{32}{3}\pi$

However, we use it to help set up the 1st moment of the volume:

$$V_z = \iiint_S z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^4 \rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\phi\,d\theta = \Big[\theta\Big]_0^{\pi/2}\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_0^4 = \frac{\pi}{2}\frac{1}{2}4^3 = 16\pi$$

So the z-component of the centroid is $\bar{z} = \frac{V_z}{V} = \frac{16\pi}{1}\frac{3}{32\pi} =$ $\underline{\;\frac{3}{2}\;}$

9. Compute $\iint_C \vec{\nabla}\cdot\vec{F}\,dS$ over the cylindrical surface $x^2 + y^2 = 4$ for $0 \le z \le 3$ if $\vec{F} = \langle xz, yz, z^2\rangle$.
You can parametrize the surface as $\vec{R}(\theta,z) = \langle 2\cos\theta, 2\sin\theta, z\rangle$.

$\iint_C \vec{\nabla}\cdot\vec{F}\,dS =$ ______________

Solution: $\vec{\nabla}\cdot\vec{F} = z + z + 2z = 4z$

	$\hat{\imath}$	$\hat{\jmath}$	$\hat{k}$	
$\vec{e}_\theta =$	$(-2\sin\theta,$	$2\cos\theta,$	$0\;)$	$\vec{N} = \hat{\imath}(2\cos\theta) - \hat{\jmath}(-2\sin\theta) + \hat{k}(0)$ $= \langle 2\cos\theta, 2\sin\theta, 0\rangle$
$\vec{e}_z =$	$(0,$	$0,$	$1\;)$	$\lvert\vec{N}\rvert = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$

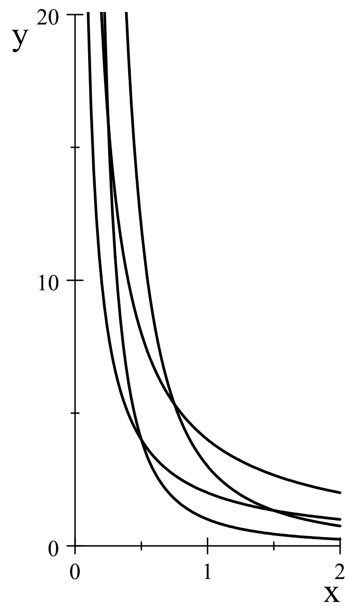
$$\iint_C \vec{\nabla}\cdot\vec{F}\,dS = \iint 4z\,dS = \iint 4z\lvert\vec{N}\rvert\,d\theta\,dz = \int_0^3\int_0^{2\pi} 4z2\,d\theta\,dz = 8\pi\Big[z^2\Big]_0^3 = \underline{\;72\pi\;}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (20 points) Compute $\iint_D x^3y^2\,dA$ over the diamond shaped region in the first quadrant bounded by the curves

$$y = \frac{2}{x} \qquad y = \frac{4}{x} \qquad y = \frac{1}{x^2} \qquad y = \frac{3}{x^2}$$

HINT: Let $u = xy$ and $v = x^2y$. What are $\frac{u}{v}$ and $\frac{u^2}{v}$?



Solution: Let $u = xy$ and $v = x^2y$. Then the boundaries are:

$$u = xy = 2 \qquad u = xy = 4 \qquad v = x^2y = 1 \qquad v = x^2y = 3$$

Notice $\frac{v}{u} = \frac{x^2y}{xy} = x$ and $\frac{u^2}{v} = \frac{x^2y^2}{x^2y} = y$.

So the position vector is

$$(x,y) = \vec{R}(u,v) = \left(\frac{v}{u}, \frac{u^2}{v}\right).$$

The Jacobian determinant is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{v}{u^2} & \frac{2u}{v} \\ \frac{1}{u} & -\frac{u^2}{v^2} \end{vmatrix} = \frac{vu^2}{u^2v^2} - \frac{2u}{uv} = \frac{1}{v} - \frac{2}{v} = -\frac{1}{v}$$

So the Jacobian factor is $J = \left|-\frac{1}{v}\right| = \frac{1}{v}$

The integrand is $x^3y^2 = \left(\frac{v}{u}\right)^3\left(\frac{u^2}{v}\right)^2 = \frac{v^3u^4}{u^3v^2} = uv$. So the integral is

$$\iint_D x^3y^2\,dA = \iint_D x^3y^2 J\,dx\,dy = \iint_D uv\frac{1}{v}\,du\,dv = \int_1^3\int_2^4 u\,du\,dv = \Big[v\Big]_1^3\left[\frac{u^2}{2}\right]_2^4 = (2)(6) = 12$$

11. (20 points) Compute the surface integral $\iint_S \vec{F}\cdot d\vec{S}$ for the vector field $\vec{F} = \langle xz, yz, z^2\rangle$ over the hemisphere $x^2+y^2+z^2 = 9$ for $z \ge 0$ with the outward orientation.
The hemisphere may be parametrized as $\vec{R} = \langle 3\sin\phi\cos\theta, 3\sin\phi\sin\theta, 3\cos\phi\rangle$.
HINT: Successively find $\vec{e}_\phi$, $\vec{e}_\theta$, $\vec{N}$, $\vec{F}\big|_{\vec{R}}$ and $\vec{F}\cdot\vec{N}$.

Solution:

$$\begin{array}{rccc} & \hat\imath & \hat\jmath & \hat k \\ \vec{e}_\phi = & (3\cos\phi\cos\theta, & 3\cos\phi\sin\theta, & -3\sin\phi\ ) \\ \vec{e}_\theta = & (-3\sin\phi\sin\theta, & 3\sin\phi\cos\theta, & 0\ ) \end{array}$$

$$\vec{N} = \hat\imath(9\sin^2\phi\cos\theta) - \hat\jmath(-9\sin^2\phi\sin\theta) + \hat k(9\sin\phi\cos\phi\cos^2\theta + 9\sin\phi\cos\phi\sin^2\theta)$$
$$= \langle 9\sin^2\phi\cos\theta, 9\sin^2\phi\sin\theta, 9\sin\phi\cos\phi\rangle$$

In the first octant, all 3 components of $\vec{N}$ are positive. So $\vec{N}$ is correctly oriented outward.

$$\vec{F}\big|_{\vec{R}} = \langle xz, yz, z^2\rangle = \langle 9\sin\phi\cos\phi\cos\theta, 9\sin\phi\cos\phi\sin\theta, 9\cos^2\phi\rangle$$

$$\vec{F}\cdot\vec{N} = 81\sin^3\phi\cos\phi\cos^2\theta + 81\sin^3\phi\cos\phi\sin^2\theta + 81\sin\phi\cos^3\phi$$
$$= 81\sin\phi\cos\phi(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi) = 81\sin\phi\cos\phi(\sin^2\phi + \cos^2\phi) = 81\sin\phi\cos\phi$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{N}\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} 81\sin\phi\cos\phi\,d\phi\,d\theta = 2\pi 81\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} = 81\pi$$

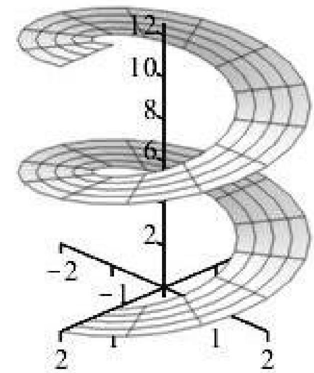
12. (20 points) A spiral ramp may be parametrized by

$\vec{R}(r,\theta) = \langle r\cos\theta, r\sin\theta, \theta\rangle$

Find the mass of the spiral ramp for $1 \le r \le 2$

and two turns, i.e. $0 \le \theta \le 4\pi$,

if the surface density is given by $\delta = \sqrt{x^2 + y^2}$.



Solution:

$$\begin{array}{cccc} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 0\ ) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 1\ ) \end{array}$$

$$\vec{N} = \hat{\imath}(\sin\theta) - \hat{\jmath}(\cos\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$

$$= \langle \sin\theta, -\cos\theta, r\rangle$$

$$\left|\vec{N}\right| = \sqrt{\sin^2\theta + \cos^2\theta + r^2} = \sqrt{1 + r^2}$$

$$\delta = \sqrt{x^2 + y^2} = r$$

$$M = \iint_R \delta\, dS = \iint_R \delta \left|\vec{N}\right| dr\, d\theta = \int_0^{4\pi} \int_1^2 r\sqrt{1 + r^2}\, dr\, d\theta = 4\pi \int_1^2 r\sqrt{1 + r^2}\, dr$$

$$u = 1 + r^2 \qquad du = 2r\,dr \qquad \frac{1}{2}du = r\,dr$$

$$M = 2\pi \int_2^5 \sqrt{u}\, du = 2\pi\left[\frac{2u^{3/2}}{3}\right]_2^5 = \frac{4\pi}{3}(5^{3/2} - 2^{3/2})$$