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MATH 221 Exam 2 Spring 2023

Section 501 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Its radius is measured to be $r = 2 \pm .02$ cm and its height is measured to be $h = 6 \pm .03$ cm.
Using the linear approximation, we compute $V = 8\pi \pm \Delta V$ where $\Delta V =$

a. 0.6π

b. 0.4π

c. 0.3π

d. 0.2π Correct

e. 0.1π

Solution: The linear approximation says

$$\Delta V = \frac{\partial V}{\partial r}\Delta r + \frac{\partial V}{\partial h}\Delta h = \frac{2}{3}\pi r h \Delta r + \frac{1}{3}\pi r^2 \Delta h = \frac{2}{3}\pi(2)(6)(.02) + \frac{1}{3}\pi(2)^2(.03) = 0.2\pi$$

2. The function $f = xy + \frac{3}{x} - \frac{9}{y}$ has a critical point at $(x,y) = (-1,3)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum

b. Local Maximum Correct

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $f_x = y - \frac{3}{x^2}$ $\quad f_y = x + \frac{9}{y^2}$

$f_{xx} = \frac{6}{x^3}$ $\quad f_{yy} = -\frac{18}{y^3}$ $\quad f_{xy} = 1$

$f_{xx}(-1,3) = -6$ $\quad f_{yy}(-1,3) = -\frac{2}{3}$ $\quad f_{xy}(-1,3) = 1$ $\quad D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3$

$D > 0$ $\quad f_{xx} < 0$ Local Maximum

3. Find the plane tangent to the graph of $z = xe^y$ at the point $(3,0)$. Its z-intercept is

a. $-e$

b. -2

c. 0 Correct

d. 2

e. e

SOLUTION:

$f = xe^y$ $\quad f(3,0) = 3$ $\quad z = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0)$

$f_x = e^y$ $\quad f_x(3,0) = 1$ $\quad = 3 + 1(x-3) + 3y$

$f_y = xe^y$ $\quad f_y(3,0) = 3$ $\quad$ When $x = y = 0$, we have $z = 3 + (-3) = 0$.

4. Find the plane tangent to the graph of $xz^3 + zy^2 + yx^4 = 8$ at the point $(1,0,2)$. Its z-intercept is

a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. $\frac{4}{3}$

d. $\frac{8}{3}$ Correct

e. 32

SOLUTION: $F(x,y,z) = xz^3 + zy^2 + yx^4$ $\quad \vec{\nabla}F = \langle z^3 + 4yx^3, 2zy + x^4, 3xz^2 + y^2 \rangle$

$\vec{N} = \left|\vec{\nabla}F\right|_{(1,0,2)} = \langle 8,1,12 \rangle$ $\quad \vec{N} \cdot X = \vec{N} \cdot P$ $\quad 8x + y + 12z = 8 \cdot 1 + 1 \cdot 0 + 12 \cdot 2 = 32$

When $x = y = 0$, we have the z-intercept $z = \frac{32}{12} = \frac{8}{3}$.

5. Sidney says the Hessian of $f(x,y,z) = x\sin y + y\cos x$ is

$$\begin{pmatrix} f_{xx} & f_{yx} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -y\cos x & \sin x - \cos y \\ \cos y - \sin x & -x\sin y \end{pmatrix}$$

Which entry is wrong?

a. f_{xx}

b. f_{yx} Correct

c. f_{xy}

d. f_{yy}

e. None of them.

Solution: $f_x = \dfrac{\partial}{\partial x}(x\sin y + y\cos x) = \sin y - y\sin x$ $\qquad f_y = \dfrac{\partial}{\partial y}(x\sin y + y\cos x) = x\cos y + \cos x$

$$f_{xx} = \frac{\partial^2}{\partial x^2}(x\sin y + y\cos x) = -y\cos x \qquad f_{yx} = \frac{\partial^2}{\partial x\partial y}(x\sin y + y\cos x) = \cos y - \sin x$$

$$f_{xy} = \frac{\partial^2}{\partial y\partial x}(\cos y + y\cos x) = \cos y - \sin x \qquad f_{yy} = \frac{\partial^2}{\partial y^2}(x\sin y + y\cos x) = -x\sin y$$

So f_{yx} is wrong. You should have known it was either f_{yx} or f_{xy} because they have to be equal.

6. If $\vec{F} = (yz, -xz, z^2)$, compute $\vec{F} \cdot \vec{\nabla} \times \vec{F}$.

a. $-2z^3$ Correct

b. z^3

c. $z^3 + xyz$

d. $-2z^3 + 2xyz$

e. 0

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & z^2 \end{vmatrix} = \hat{\imath}(x) - \hat{\jmath}(-y) + \hat{k}(-z - z) = (x, y, -2z)$

$\vec{F} \cdot \vec{\nabla} \times \vec{F} = yzx - xzy - z^2 2z = -2z^3$

7. Find the point (x,y) at which the divergence of $\vec{F} = \langle 6x^2 - xy^2, -y^2 - 2x^2 y\rangle$ is a maximum.

a. $(3,1)$

b. $(-3,1)$

c. $(-3,-1)$

d. $(3,-1)$ Correct

e. $(0,0)$

Solution: The divergence is $D = \vec{\nabla} \cdot \vec{F} = 12x - y^2 - 2y - 2x^2$.

To find its maximum, we set its derivatives equal to 0 and solve:

$$D_x = 12 - 4x = 0 \qquad x = 3 \qquad\qquad D_y = -2y - 2 = 0 \qquad y = -1$$

The point is $(x,y) = (3,-1)$. It has to be a maximum because D is a parabola opening down.

8. Find the mass of a wire in the shape of the semi-circle $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta)$ for $0 \le \theta \le \pi$ if the linear density is $\delta = y$.

a. 2π
b. 8π
c. 8
d. 16
e. 32 Correct

Solution: The tangent vector is $\vec{v} = (-4\sin\theta, 4\cos\theta)$ and its length is $|\vec{v}| = \sqrt{16\sin^2\theta + 16\cos^2\theta} = 4$.
The density along the curve is $\delta(\vec{r}(t)) = y = 4\sin\theta$. So the mass is:
$M = \int_0^\pi \delta\, ds = \int_0^\pi \delta(\vec{r}(t))\,|\vec{v}|\,d\theta = \int_0^\pi 4\sin\theta\, 4\, d\theta = \Big[-16\cos\theta\Big]_0^\pi = 16 - -16 = 32.$

9. A bead is pushed along a wire in the shape of the twisted cubic $\vec{r}(t) = (t^3, t^2, t)$ by the force $\vec{F} = \langle z^3, yz^2, xz^2\rangle$ from $(1,1,1)$ to $(8,4,2)$. Find the work done.

a. 186
b. $\frac{384}{7}$
c. $\frac{381}{7}$
d. 63 Correct
e. 64

Solution: $\vec{v} = \langle 3t^2, 2t, 1\rangle$ $\quad \vec{F}(\vec{r}(t)) = \langle t^3, t^4, t^5\rangle$ $\quad \vec{F}\cdot\vec{v} = 3t^5 + 2t^5 + t^5 = 6t^5$
$W = \int \vec{F}\cdot d\vec{s} = \int_1^2 \vec{F}\cdot\vec{v}\,dt = \int_1^2 6t^5\,dt = [t^6]_1^2 = 64 - 1 = 63$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. $(12$ points$)$ Find the point $P = (x,y,z)$ on the plane $x + y - z = 2$ which is closest to the point $Q = (1,0,2)$. Find the distance from P to Q.

Solution: Minimize the distance $D = \sqrt{(x-1)^2 + y^2 + (z-2)^2}$ or its square:

$f = D^2 = (x-1)^2 + y^2 + (z-2)^2$ subject to the constraint $z = x + y - 2$.

So $f = (x-1)^2 + y^2 + (x+y-4)^2$

$f_x = 2(x-1) + 2(x+y-4) = 0 \quad \Rightarrow \quad 4x + 2y - 10 = 0 \quad \Rightarrow \quad 2x + y = 5$ (a)

$f_y = 2y + 2(x+y-4) = 0 \quad \Rightarrow \quad 2x + 4y - 8 = 0 \quad \Rightarrow \quad x + 2y = 4$ (b)

(a)-2*(b): $-3y = -3 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad x = 2 \quad \Rightarrow \quad z = 1$

$P = (2,1,1)$

$D = \sqrt{(2-1)^2 + 1^2 + (1-2)^2} = \sqrt{3}$

11. $(12$ points$)$ As Duke Skywater flies the Centurion Eagle through the galaxy he wants to maximize the Power of the Force which is given by $F = \dfrac{1}{D}$ where D is the dark matter density given by $D = x^3 + y^3 + z^3$.

If his current position is $\vec{r} = (2,1,1)$ and his current velocity is $\vec{v} = (0.5,-0.2,-0.8)$, what is the current rate of change of the Power of the Force, $\dfrac{dF}{dt}$?

(Plug in numbers but you don't need to simplify.)

Solution: The position says $x = 2, \quad y = 1, \quad z = 1$.

The velocity says $\dfrac{dx}{dt} = 0.5, \quad \dfrac{dy}{dt} = -0.2, \quad \dfrac{dz}{dt} = -0.8$.

Currently, $D = x^3 + y^3 + z^3 = 2^3 + 1^3 + 1^3 = 10$.

We use the chain rule twice:

$$\frac{dF}{dt} = \frac{dF}{dD}\frac{dD}{dt} = \frac{dF}{dD}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) = \frac{-1}{D^2}\left(3x^2\frac{dx}{dt} + 3y^2\frac{dy}{dt} + 3z^2\frac{dz}{dt}\right)$$

$$= \frac{-1}{10^2}(3 \cdot 4 \cdot (0.5) + 3 \cdot 1 \cdot (-0.2) + 3 \cdot 1 \cdot (-0.8)) = -\frac{3}{100} = -0.03$$

12. (14 points) Determine whether or not each of these limits exists. If it exists, find its value.

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y^2}{x^6+3y^3}$

SOLUTION: Straight line approaches: $y = mx$

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{3x^2y^2}{x^6+3y^3} = \lim_{x\to 0} \frac{3x^2m^2x^2}{x^6+3m^3x^3} = \lim_{x\to 0} \frac{3m^2x}{x^3+3m^3} = \frac{0}{3m^3} = 0$$

Quadratic approaches: $y = mx^2$

$$\lim_{\substack{y=mx^2 \\ x\to 0}} \frac{3x^2y^2}{x^6+3y^3} = \lim_{x\to 0} \frac{3x^2m^2x^4}{x^6+3m^3x^6} = \lim_{x\to 0} \frac{3m^2}{1+3m^3} = \frac{3m^2}{1+3m^3} \neq 0 \qquad \text{if } m \neq 0.$$

Limit does not exist because these are different.

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

SOLUTION: Switch to polar: $x = r\cos\theta \quad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} \frac{r\cos\theta\, r^2\sin^2\theta}{r^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} r\cos\theta\sin^2\theta = 0$$

because $r \to 0$ while $\cos\theta\sin^2\theta$ is bounded: $-1 \le \cos\theta\sin^2\theta \le 1$.

13. (12 points) Find a scalar potential, f, for the vector field $\vec{F} = \langle \cos y, \sin z - x\sin y, 2z + y\cos z\rangle$. (You MUST SHOW your derivation.)

Solution: We need to find a function $f(x,y,z)$ satisfying $\vec{\nabla} f = \vec{F} = \langle \cos y, \sin z - x\sin y, 2z + y\cos z\rangle$. Or:

$$(1)\quad \partial_x f = \cos y \qquad (2)\quad \partial_y f = \sin z - x\sin y \qquad (3)\quad \partial_z f = 2z + y\cos z$$

Equation (1) says: $f = x\cos y + g(y,z)$ Then $\partial_y f = -x\sin y + \partial_y g$.

Comparing to equation (2) says: $\partial_y g = \sin z$.

So $g = y\sin z + h(z)$ and so $f = x\cos y + y\sin z + h(z)$. Then $\partial_z f = y\cos z + h'(z)$.

Comparing to equation (3) says: $h'(z) = 2z$

Therefore $h = z^2 + C$ and so $f = x\cos y + y\sin z + z^2 + C$