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MATH 221 Final Spring 2023

Section 501 Solutions P. Yasskin

1-9	/36	12	/15
10	/15	13	/25
11	/15	Total	/106

Multiple Choice: (4 points each. No part credit.)

1. Find the angle between the line $\vec{r}(t) = (3+2t, 2, 5+2t)$ and the normal to the plane $x+y+2z = 4$.

a. $\frac{\pi}{6}$ Correct

b. $\frac{\pi}{4}$

c. $\frac{\pi}{3}$

d. $\frac{\pi}{2}$

e. $\frac{2\pi}{3}$

Solution: The direction of the line is $\vec{v} = \langle 2,0,2 \rangle$. The normal to the plane is $\vec{N} = \langle 1,1,2 \rangle$.
So the angle between them satisfies:

$$\cos\theta = \frac{\vec{v} \cdot \vec{N}}{|\vec{v}||\vec{N}|} = \frac{2+0+4}{\sqrt{4+4}\sqrt{1+1+4}} = \frac{6}{\sqrt{8}\sqrt{6}} = \frac{2 \cdot 3}{2\sqrt{2}\sqrt{2}\sqrt{3}} = \frac{\sqrt{3}}{2} \qquad \theta = \frac{\pi}{6}$$

2. Find the equation of the plane tangent to $z = x^2y + y^2x$ at the point $(x,y) = (1,2)$.
Which of the following points lies on the tangent plane?

a. $(2,1,19)$

b. $(2,1,9)$ Correct

c. $(3,3,17)$

d. $(3,3,21)$

Solution: $f = x^2y + y^2x \qquad f_x = 2xy + y^2 \qquad f_y = x^2 + 2yx$

$f(1,2) = 2+4 = 6 \qquad f_x(1,2) = 4+4 = 8 \qquad f_y(1,2) = 1+4 = 5$

The tangent plane is $z = f_{\tan}(x,y) = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) = 6 + 8(x-1) + 5(y-2)$

$z = f_{\tan}(2,1) = 6 + 8(2-1) + 5(1-2) = 6+8-5 = 9 \qquad (2,1,9)$

$z = f_{\tan}(3,3) = 6 + 8(3-1) + 5(3-2) = 6+16+5 = 27 \qquad (3,3,27)$

3. Find the plane tangent to the surface $x^2z + y^2z + xyz = 21$ at the point $P = (1,2,3)$.
Find the z-intercept.

a. $z = 3$

b. $z = 5$

c. $z = 7$

d. $z = 9$ Correct

e. $z = 11$

Solution: Let $f = x^2z + y^2z + xyz$. So $\vec{\nabla} f = \langle 2xz + yz, 2yz + xz, x^2 + y^2 + xy \rangle$.
The normal is $\vec{N} = \vec{\nabla} f \big|_P = \langle 6+6, 12+3, 1+4+2 \rangle = \langle 12, 15, 7 \rangle$.
The plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $12x + 15y + 7z = 12(1) + 15(2) + 7(3) = 63$
The z-intercept satisfies $x = 0$ and $y = 0$. So $7z = 63$ or $z = 9$.

4. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. A cone currently has radius $r = 5$ cm and height $h = 8$ cm.
If the radius decreases at $0.3 \frac{\text{cm}}{\text{sec}}$ while the volume decreases by $8\pi \frac{\text{cm}^3}{\text{sec}}$,
find the rate at which the height is currently changing. $\frac{dh}{dt} =$

a. $\frac{3}{25} \frac{\text{cm}}{\text{sec}}$

b. $\frac{48}{25} \frac{\text{cm}}{\text{sec}}$

c. $-\frac{25}{3} \frac{\text{cm}}{\text{sec}}$

d. $-\frac{25}{48} \frac{\text{cm}}{\text{sec}}$

e. $0 \frac{\text{cm}}{\text{sec}}$ Correct

Solution: By chain rule, $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}$.
We plug in the numbers: $r = 5$, $h = 8$, $\frac{dr}{dt} = -0.3$ and $\frac{dV}{dt} = -8\pi$. Then solve for $\frac{dh}{dt}$.
$-8\pi = \frac{2}{3}\pi(5)(8)(-0.3) + \frac{1}{3}\pi(5)^2\frac{dh}{dt}$ $\quad \frac{25}{3}\pi\frac{dh}{dt} = -8\pi + 8\pi = 0$ $\quad \frac{dh}{dt} = 0$.

5. The function $f(x,y) = x^4 - 8xy + \frac{1}{16}y^4$ has a critical point at $(2,4)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum   Correct

b. Local Maximum

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $f_x = 4x^3 - 8y \qquad f_y = -8x + \frac{1}{4}y^3 \qquad f_{xx} = 12x^2 \qquad f_{yy} = \frac{3}{4}y^2 \qquad f_{xy} = -8$

$f_{xx}(2,4) = 12 \cdot 4 = 48 \qquad f_{yy}(2,4) = \frac{3}{4}16 = 12 \qquad f_{xy}(2,4) = -8$

$D = f_{xx}f_{yy} - {f_{xy}}^2 = 48 \cdot 12 - 8^2 = 512$

Since $D > 0$ and $f_{xx} > 0$ the point is a local minimum.

6. Compute $\int_0^8 \int_{x^{1/3}}^2 \cos(y^4)\,dy\,dx$

HINT: Reverse the order of integration.

a. $\frac{1}{4}\sin(4) - \frac{1}{4}$

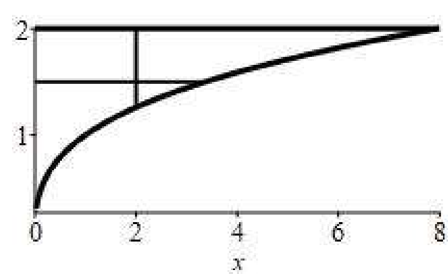
b. $\frac{1}{4}\sin(64) - \frac{1}{4}$

c. $\frac{1}{4}\sin(64)$

d. $\frac{1}{4}\sin(16) - \frac{1}{4}$

e. $\frac{1}{4}\sin(16)$ Correct

Solution: Plot the region. Reverse the order.



Compute new limits: $y = x^{1/3} \quad \Rightarrow \quad x = y^3$

$$\int_0^8 \int_{x^{1/3}}^2 \cos(y^4)\,dy\,dx = \int_0^2 \int_0^{y^3} \cos(y^4)\,dx\,dy = \int_0^2 \cos(y^4)\big[x\big]_{x=0}^{y^3} dy$$

$$= \int_0^2 y^3 \cos(y^4)\,dy = \left[\frac{\sin(y^4)}{4}\right]_{y=o}^2 = \frac{1}{4}\sin(16)$$

7. Consider the parametric surface $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

Find the normal line at the point $P = \vec{R}\left(\sqrt{2}, \frac{\pi}{4}\right) = (1,1,2)$.

It intersects the xy-plane at

a. $(-3,-3,0)$

b. $(-3,-3,4)$

c. $(5,5,0)$ Correct

d. $(5,5,4)$

e. $\left(2\sqrt{2}, 2\sqrt{2}, 0\right)$

Solution: We find the normal: $\vec{e}_r = (\cos\theta, \sin\theta, 2r)$ $\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 0)$

$\vec{N} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$ At P: $\vec{N}\Big|_P = \left(-2 \cdot 2\frac{1}{\sqrt{2}}, -2 \cdot 2\frac{1}{\sqrt{2}}, \sqrt{2}\right) = \left(-2\sqrt{2}, -2\sqrt{2}, \sqrt{2}\right)$

The normal line is $X = P + t\vec{N} = \left(1 - 2\sqrt{2}\,t, 1 - 2\sqrt{2}\,t, 2 + \sqrt{2}\,t\right)$. The line intersects the xy-plane when $z = 2 + \sqrt{2}\,t = 0$. So $t = -\sqrt{2}$. Then $X = \left(1 + 2\sqrt{2}\sqrt{2}, 1 + 2\sqrt{2}\sqrt{2}, 2 - \sqrt{2}\sqrt{2}\right) = (5,5,0)$.

8. On Exam 3, you solved the problem:

"Given the function $f(x,y,z) = xy + 3z$ compute the vector line integral $\int_A^B \vec{\nabla} f \cdot d\vec{s}$ along the twisted cubic $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $A = \left(1, 1, \frac{2}{3}\right)$ and $B = (3,9,18)$."

You can now do it more easily using a Theorem. Which Theorem?

a. Fundamental Theorem of Calculus for Curves Correct

b. Green's Theoprem

c. 2D Stokes' Theorem

d. Stokes' Theorem

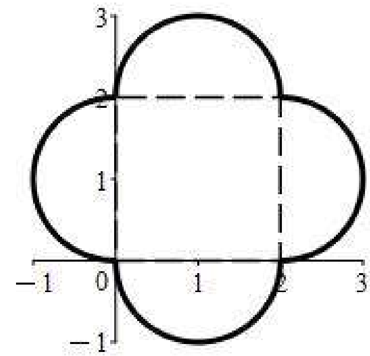
e. Gauss' Theorem

Solution: The Fundamental Theorem of Calculus for Curves says

$$\int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = [(3)(9) + 3(18)] - \left[(1)(1) + 3\left(\frac{2}{3}\right)\right] = 78$$

9. Compute the line integral $\oint (3y + \cos x)\,dx + (5x - \sin y)\,dy$ counterclockwise around the boundary of the region shown consisting of a square and 4 semicircles.

HINT: Use a Theorem.



a. $4 + 2\pi$

b. $1 + 2\pi$

c. $8 + 4\pi$ Correct

d. $\pi + 2\pi^2$

e. $2\pi + 4\pi^2$

Solution: We apply Green's Theorem with $P = 3y + \cos x$ and $Q = 5x - \sin y$:

$$\oint (3y + \cos x)\,dx + (5x - \sin y)\,dy = \oint P\,dx + Q\,dy = \iint (\partial_x Q - \partial_y P)\,dx\,dy = \iint (5 - 3)\,dx\,dy$$

$$= 2\,\text{Area} = 2\left(4 + 4 \times \frac{1}{2}\pi 1^2\right) = 8 + 4\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (15 points) Find the volume of the largest rectangular solid with 3 faces in the coordinate planes and the opposite vertex on the plane $\frac{x}{9} + \frac{y}{6} + \frac{z}{3} = 1$.

Solution: We maximize $V = xyz$ where (x,y,z) is the vertex on the plane $g = \frac{x}{9} + \frac{y}{6} + \frac{z}{3} = 1$.
The gradients are: $\vec{\nabla} V = \langle yz, xz, xy \rangle$ and $\vec{\nabla} g = \left\langle \frac{1}{9}, \frac{1}{6}, \frac{1}{3} \right\rangle$. The Lagrange equations are

$$yz = \frac{1}{9}\lambda \qquad xz = \frac{1}{6}\lambda \qquad xy = \frac{1}{3}\lambda \qquad \text{or} \qquad 9yz = 6xz = 3xy \qquad \text{or} \qquad x = \frac{3}{2}y \qquad z = \frac{1}{2}y$$

We plug into the plane: $\frac{y}{6} + \frac{y}{6} + \frac{y}{6} = 1$ or $y = 2$ So $x = 3$ $z = 1$

So $V = xyz = 3 \cdot 2 \cdot 1 = 6$

11. (15 points) Consider the parametric surface $\vec{R}(u,v) = \left(u^2, v^2, \sqrt{2}\,uv\right)$
for $0 \le u \le 2$ and $0 \le v \le 3$.
Find the mass of the surface if the surface density is $\delta = \frac{1}{x+y}$.
HINT: Factor out a $\sqrt{8}$.

Solution: The tangent and normal vectors and the length of the normal are:

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (2u, & 0, & \sqrt{2}\,v) \\ \vec{e}_v = & (0, & 2v, & \sqrt{2}\,u) \end{vmatrix} \qquad \vec{N} = \hat{\imath}\left(-2\sqrt{2}\,v^2\right) - \hat{\jmath}\left(2\sqrt{2}\,u^2\right) + \hat{k}(4uv) = \left(-2\sqrt{2}\,v^2, -2\sqrt{2}\,u^2, 4uv\right)$$

$$\left|\vec{N}\right| = \sqrt{8v^4 + 8u^4 + 16u^2v^2} = \sqrt{8}\sqrt{v^4 + u^4 + 2u^2v^2} = \sqrt{8}\sqrt{(v^2 + u^2)^2} = \sqrt{8}\,(v^2 + u^2)$$

The density is $\delta = \frac{1}{x+y} = \frac{1}{u^2 + v^2}$. So the mass is:

$$M = \iint \delta\, dS = \int_0^3 \int_0^2 \delta \left|\vec{N}\right| du\, dv = \int_0^3 \int_0^2 \frac{1}{u^2 + v^2} \sqrt{8}\,(v^2 + u^2)\, du\, dv = \sqrt{8} \int_0^3 \int_0^2 du\, dv = 12\sqrt{2}$$

12. $\left(15 \text{ points}\right)$ Given the vector field $\vec{F}(x,y,z) = \langle yz^2, -xz^2, z^3 \rangle$ compute the vector surface integral $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ along the side surface of the cylinder $x^2 + y^2 = 4$ for $2 \le z \le 6$, oriented **outward**. (There are no ends on the cylinder.) On Exam 3, you solved this directly.
Now solve it using Stokes' Theorem, using the following steps.

a. Compute the line integral $\int_{z=6} \vec{F} \cdot d\vec{s}$ around the circle $x^2 + y^2 = 4$ for $z = 6$, **counterclockwise** as seen from above.

The circle may be parametrized by $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 6)$.

The velocity is $\vec{v} =$

On the circle $\vec{F}\Big|_{\vec{r}(\theta)} =$

$\int_{z=6} \vec{F} \cdot d\vec{s} =$

SOLUTION: $\vec{v} = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$ $\quad \vec{F}\Big|_{\vec{r}(\theta)} = \langle 72\sin\theta, -72\cos\theta, 216 \rangle$

$$\int_{z=6} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -144\, d\theta = -288\pi$$

b. Compute the line integral $\int_{z=6} \vec{F} \cdot d\vec{s}$ around the circle $x^2 + y^2 = 4$ for $z = 2$, **counterclockwise** as seen from above.

The circle may be parametrized by $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$.

The velocity is $\vec{v} =$

On the circle $\vec{F}\Big|_{\vec{r}(\theta)} =$

$\int_{z=2} \vec{F} \cdot d\vec{s} =$

SOLUTION: $\vec{v} = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$ $\quad \vec{F}\Big|_{\vec{r}(\theta)} = \langle 8\sin\theta, -8\cos\theta, 8 \rangle$

$$\int_{z=2} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -16\, d\theta = -32\pi$$

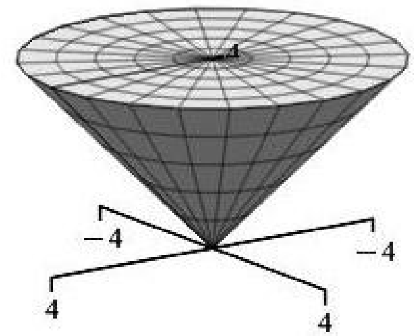
c. Combine the answers to parts (a) and (b) (justifying your orientations) to find

$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} =$

SOLUTION: Since C is oriented outward, the upper circle must be clockwise and the lower circle must be counterclockwise. So we put a minus before the integral for $z = 6$:

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = -\int_{z=6} \vec{F} \cdot d\vec{s} + \int_{z=2} \vec{F} \cdot d\vec{s} = --288\pi - 32\pi = 256\pi$$

13. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = \langle yz^2, xz^2, z(x^2+y^2)\rangle$ and the solid cone $\sqrt{x^2+y^2} \le z \le 4$



Be sure to check orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence of $\vec{F}$:

Solution: $\vec{\nabla} \cdot \vec{F} = 0 + 0 + x^2 + y^2 = r^2$

b. Compute the left hand side:

Solution: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^4\int_r^4 r^2\, r\, dz\, dr\, d\theta = 2\pi \int_0^4 [r^3 z]_{z=r}^4\, dr = 2\pi \int_0^4 (4r^3 - r^4)\, dr$

$$= 2\pi\left[r^4 - \frac{r^5}{5}\right]_0^4 = 2\pi 4^4\left[1 - \frac{4}{5}\right] = \frac{512}{5}\pi$$

Second the Right Hand Side: The boundary surface consists of a disk and a cone.

Disk:

c. Parametrize the disk.

Solution: $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 4)$

d. Compute the tangent vectors:

Solution: $\vec{e}_r = \langle \cos\theta, \quad \sin\theta, \quad 0\rangle$

$\vec{e}_\theta = \langle -r\sin\theta, \quad r\cos\theta, \quad 0\rangle$

e. Compute the normal vector:

Solution: $\vec{N} = \langle 0, 0, r\rangle$ This points up as required.

f. Evaluate $\vec{F} = \langle yz^2, xz^2, z(x^2+y^2)\rangle$ on the disk:

Solution: $\vec{F}\big|_{\vec{R}(r,\theta)} = \langle 16r\sin\theta, 16r\cos\theta, 4r^2\rangle$

g. Compute the dot product:

Solution: $\vec{F} \cdot \vec{N} = 4r^3$

h. Compute the flux through D:

Solution: $\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^4 4r^3\, dr\, d\theta = 2\pi\left[r^4\right]_0^4 = 512\pi$

(continued)

Cone:

The cone may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

i. Compute the tangent vectors:

Solution: $\vec{e}_r = \langle \cos\theta, \quad \sin\theta, \quad 1 \rangle$

$\vec{e}_\theta = \langle -r\sin\theta, \quad r\cos\theta, \quad 0 \rangle$

j. Compute the normal vector:

Solution: $\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle -r\cos\theta, -r\sin\theta, r \rangle$

This is up and in. We need down and out.

Reverse: $\vec{N} = \langle r\cos\theta, r\sin\theta, -r \rangle$

k. Evaluate $\vec{F} = \langle yz^2, xz^2, z(x^2+y^2) \rangle$ on the cone:

Solution: $\vec{F}\Big|_{\vec{R}(r,\theta)} = \langle r^3\sin\theta, r^3\cos\theta, r^3 \rangle$

l. Compute the dot product:

Solution: $\vec{F} \cdot \vec{N} = r^4\sin\theta\cos\theta + r^4\sin\theta\cos\theta - r^4 = r^4(2\sin\theta\cos\theta - 1)$

m. Compute the flux through C:

Solution: $\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^4 r^4(2\sin\theta\cos\theta - 1)\, dr\, d\theta$

$= \left[\sin^2\theta - \theta \right]_0^{2\pi} \left[\frac{r^5}{5} \right]_0^4 = -2\pi\frac{4^5}{5} = -\frac{2048}{5}\pi$

n. Compute the **TOTAL** right hand side:

Solution: $\iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} + \iint_C \vec{F} \cdot d\vec{S} = 512\pi - \frac{2048}{5}\pi = \frac{512}{5}\pi$

which agrees with (b).