Name________________________________ Section:__________

1-9	/54	12	/10
10	/20	13	/10
11	/10	Total	/104

MATH 221 Exam 1, Version A Fall 2023

502,503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. The circle $(x-2)^2+(y-5)^2=9$ is tangent to which line?
HINT: Draw a picture.

a. $x=1$

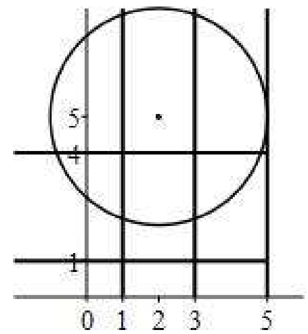
b. $x=3$

c. $x=5$ Correct

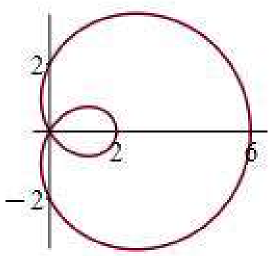
d. $y=1$

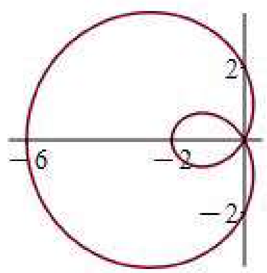
e. $y=4$

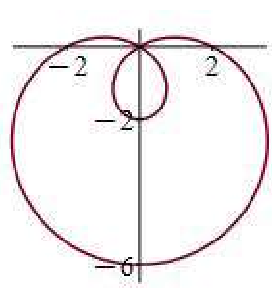
Solution: See the plot.

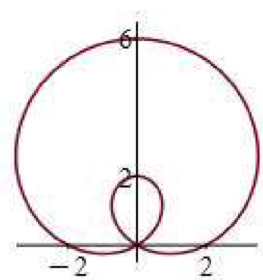


2. Which of the following is the plot of the polar curve $r=2-4\cos\theta$?

a.


b.


c.


d.


Correct

Solution: $r(0)=2-4(1)=-2$ (measured left since $r<0$)
$r(\pi)=2-4(-1)=6$ (measured left since θ is left).

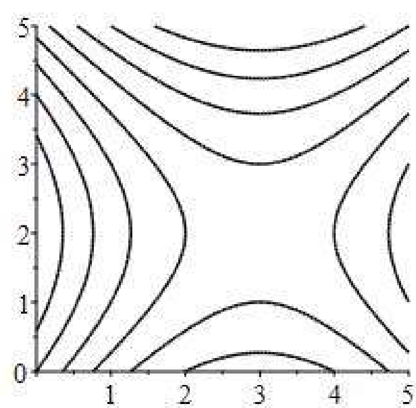
3. The plot at the right is the contour plot of which function?

a. $z=(x-2)^2+(y-3)^2$

b. $z=(x-2)^2-(y-3)^2$

c. $z=(x-3)^2+(y-2)^2$

d. $z=(x-3)^2-(y-2)^2$ Correct



Solution: The plot is centered at $(3,2)$. So the function is $z=\pm(x-3)^2\pm(y-2)^2$.
Since the contours are hyperbolas, not circles, the function is $z=(x-3)^2-(y-2)^2$.

4. The force $\vec{F} = \langle 5,2 \rangle$ pushes a mass from $P = (5,4)$ to $Q = (12,1)$.
Find the angle between the force and the displacement.

- **a**. $30°$
- **b**. $45°$ Correct
- **c**. $60°$
- **d**. $120°$
- **e**. $135°$

Solution: The displacement is $\vec{D} = Q - P = \langle 7,-3 \rangle$. So the angle satisfies:

$$\cos\theta = \frac{\vec{F} \cdot \vec{D}}{|\vec{F}|\,|\vec{D}|} = \frac{35-6}{\sqrt{25+4}\,\sqrt{49+9}} = \frac{29}{\sqrt{29}\,\sqrt{58}} = \frac{1}{\sqrt{2}} \qquad \theta = 45°$$

5. Find the area of the triangle with vertices $A = (1,2,3)$, $B = (4,6,4)$ and $C = (4,6,6)$.

- **a**. 1
- **b**. 4
- **c**. 5 Correct
- **d**. 8
- **e**. 10

Solution: $\overrightarrow{AB} = B - A = \langle 3,4,1 \rangle \qquad \overrightarrow{AC} = C - A = \langle 3,4,3 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 4 & 1 \\ 3 & 4 & 3 \end{vmatrix} = \hat{\imath}(12-4) - \hat{\jmath}(9-3) + \hat{k}(12-12) = \langle 8,-6,0 \rangle$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}|\langle 8,-6,0 \rangle| = \frac{1}{2}\sqrt{64+36} = \frac{1}{2}\sqrt{100} = 5$$

6. Find an equation of the line through the point $P = (2,3,4)$
which is perpendicular to the plane $4x + 3y + 2z = 15$.
Then find the point where the line passes through the xy-plane.

- **a**. $(x,y,z) = (10,9,0)$
- **b**. $(x,y,z) = (-10,-9,0)$
- **c**. $(x,y,z) = (-6,-3,0)$ Correct
- **d**. $(x,y,z) = (6,3,0)$
- **e**. $(x,y,z) = (6,6,0)$

Solution: The direction of the line is the normal to the plane, $\vec{v} = \vec{N} = \langle 4,3,2 \rangle$. The point is given.
So the line is $X = P + t\vec{v}$ or $(x,y,z) = (2,3,4) + t\langle 4,3,2 \rangle$ or $x = 2 + 4t \quad y = 3 + 3t \quad z = 4 + 2t$.
The line intersects the xy-plane when $z = 0 = 4 + 2t$ or $t = -2$. So
$(x,y,z) = (2,3,4) - 2\langle 4,3,2 \rangle = (-6,-3,0)$

7. Classify the quadratic surface: $-2x^2 + 4x + 3y^2 + 6y - z + 3 = 0$

a. elliptic paraboloid opening up in the z-direction
b. elliptic paraboloid opening down in the z-direction
c. hyperbolic paraboloid opening up in the x-direction and down in the y-direction
d. hyperbolic paraboloid opening up in the y-direction and down in the x-direction Correct
e. hyperbolic cylinder

Solution: Since the x and y terms are quadratic with opposite signs and z is linear, the surface is a hyperbolic paraboloid with axis parallel to the z-axis. To find which way it opens, we solve for z:

$$z = -2x^2 + 4x + 3y^2 + 6y + 3$$

Since the coefficient of x^2 is negative and that of y^2 is positive,
the paraboloid opens up in the y direction and down in the x direction.

8. If an airplane is flying from West to East directly above the equator, where does $\vec{B}$ point? Why?

a. North Correct
b. South
c. West
d. Up
e. Down

Solution: $\vec{T}$ points East. $\vec{N}$ points Down toward the center of the Earth.
So $\vec{B} = \vec{T} \times \vec{N}$ points North.

9. Find the circulation in a bowl of water, counterclockwise around the circle $x^2 + y^2 = 9$, with $z = 2$, if its fluid velocity field is $\vec{V} = \langle 2x - y, x + 2y, -z \rangle$.

a. 3π
b. 6π
c. 12π
d. 15π
e. 18π Correct

Solution: The circle may be parametrized by $\vec{r}(t) = (3\cos t, 3\sin t, 2)$.
Its tangent vector is $\vec{v} = \langle -3\sin t, 3\cos t, 0 \rangle$. The fluid velocity on the curve is
$\vec{V}(\vec{r}(t)) = \langle 6\cos t - 3\sin t, 3\cos t + 6\sin t, -2 \rangle$.
The dot product of the fluid velocity and the tangent vector is
$\vec{V}(\vec{r}(t)) \cdot \vec{v} = -3\sin t(6\cos t - 3\sin t) + 3\cos t(3\cos t + 6\sin t) + 0 = 9\sin^2 t + 9\cos^2 t = 9$
So the circulation is
Circ $= \oint \vec{V} \cdot d\vec{s} = \int_0^{2\pi} \vec{V}(\vec{r}(t)) \cdot \vec{v}\, dt = \int_0^{2\pi} 9\, dt = \Big[9t \Big]_0^{2\pi} = 18\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (20 pts) Consider the twisted cubic $\vec{r} = (6t, 3t^2, t^3)$. Compute each of the following.
Note: $4 + 4t^2 + t^4 = (2 + t^2)^2$

a. (6 pts) Arc length between $(0,0,0)$ and $(6,3,1)$.

Solution: $\vec{v} = \langle 6, 6t, 3t^2 \rangle \qquad |\vec{v}| = \sqrt{36 + 36t^2 + 9t^4} = 3\sqrt{4 + 4t^2 + t^4} = 3\sqrt{(2 + t^2)^2} = 3(2 + t^2)$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 3(2 + t^2)\,dt = 3\left[2t + \frac{t^3}{3}\right]_0^1 = 3\left[2 + \frac{1}{3}\right] = 7$$

b. (6 pts) Curvature $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$.
HINT: Factor out an 18^2.

Solution: $\vec{a} = \langle 0, 6, 6t \rangle \qquad \vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 6 & 6t & 3t^2 \\ 0 & 6 & 6t \end{vmatrix} = \langle 36t^2 - 18t^2, -36t, 36 \rangle = \langle 18t^2, -36t, 36 \rangle$

$$|\vec{v} \times \vec{a}| = \sqrt{18^2 t^4 + 36^2 t^2 + 36^2} = 18\sqrt{t^4 + 4t^2 + 4} = 18\sqrt{(2 + t^2)^2} = 18(2 + t^2)$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{18(2 + t^2)}{3^3(2 + t^2)^3} = \frac{2}{3(2 + t^2)^2}$$

c. (4 pts) Tangential acceleration, a_T.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

Solution: $a_T = \dfrac{d}{dt}|\vec{v}| = \dfrac{d}{dt} 3(2 + t^2) = 6t$

d. (4 pts) Normal acceleration, a_N.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

Solution: $a_n = \kappa|\vec{v}|^2 = \dfrac{2}{3(2 + t^2)^2} 3^2(2 + t^2)^2 = 6$

11. (10 pts) Find the y-component of the center of mass of the semicircle $y = \sqrt{9-x^2}$ if its linear density is $\delta(x,y) = y$.
HINT: The semicircle may be parametrized by $\vec{r}(t) = (3\cos t, 3\sin t)$ for $0 \le t \le \pi$.

Solution: The velocity is $\vec{v} = \langle -3\sin t, 3\cos t\rangle$. The speed is $|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t} = 3$.
The density along the curve is $\delta = y = 3\sin t$. The mass is
$M = \int_0^\pi \delta|\vec{v}|\,dt = \int_0^\pi 3\sin t\, 3\,dt = \Big[-9\cos t\Big]_0^\pi = 9 - -9 = 18$
The y-moment is
$M_y = \int_0^\pi y\delta|\vec{v}|\,dt = \int_0^\pi 3\sin t\, 3\sin t\, 3\,dt = 27\int_0^\pi \sin^2 t\,dt = 27\int_0^\pi \frac{1-\cos(2t)}{2}\,dt$
$= \frac{27}{2}\left[t - \frac{\sin(2t)}{2}\right]_0^\pi = \frac{27\pi}{2}$
So the y-component of the center of mass is $\bar{y} = \frac{M_y}{M} = \frac{27\pi}{2}\frac{1}{18} = \frac{3\pi}{4}$.

12. (10 pts) Write the vector, $\langle 1,1,4\rangle$, as a linear composition of $\langle 2,1,3\rangle$ and $\langle 3,1,2\rangle$, i.e. find a and b so that:

$$\langle 1,1,4\rangle = a\langle 2,1,3\rangle + b\langle 3,1,2\rangle$$

or show it cannot be done.

Solution: We need to solve the equations: $2a+3b = 1 \quad a+b = 1 \quad 3a+2b = 4$
The first equation minus twice the second is $\boxed{b=-1}$. Then the second equation says $\boxed{a=2}$.
We check the third equation: $3a+2b = 3(2)+2(-1) = 4$ which is correct. So:

$$\langle 1,1,4\rangle = 2\langle 2,1,3\rangle - 1\langle 3,1,2\rangle$$

13. (10 pts) Consider the line and the plane:

$$L: \quad \frac{x-2}{2} = \frac{y-3}{2} = \frac{z-1}{2}$$

$$P: \quad 4x - 3y + z = 4$$

Determine if they are parallel or intersecting. If they intersect, find the point of intersection.
You MUST show why they are or are not parallel.

Solution: The parametric equation of the line is $x = 2+2t$, $y = 3+2t$, $z = 1+2t$.
The direction of the line is $\vec{v} = \langle 2,2,2\rangle$. The normal to the plane is $\vec{N} = \langle 4,-3,1\rangle$.
Since $\vec{v}\cdot\vec{N} = 8-6+2 = 4 \ne 0$, they are not perpendicular and the line is not parallel to the plane.
We substitute the line into the plane and solve for t:
$4x-3y+z = 4(2+2t)-3(3+2t)+(1+2t) = 4 \quad \Rightarrow \quad 4t = 4 \quad \Rightarrow \quad t = 1$
We substitute back into the line to get the point of intersection:
$(x,y,z) = (2+2t, 3+2t, 1+2t) = (2+2(1), 3+2(1), 1+2(1)) = (4,5,3)$