Name________________________ Section:________

MATH 221 Exam 1, Version B Fall 2023

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Multiple Choice: (6 points each. No part credit.)

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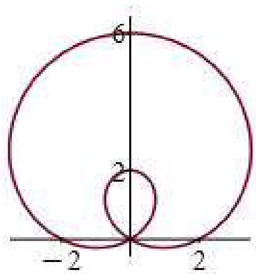
|---|---|---|---|
| 1-9 | /54 | 12 | /10 |
| 10 | /20 | 13 | /10 |
| 11 | /10 | Total | /104 |

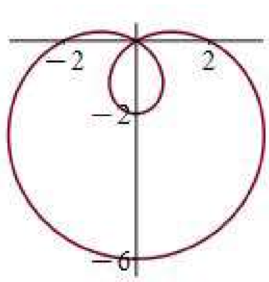
**1**. Find the sphere which is tangent to the $z$-axis whose center is $(4,3,2)$.

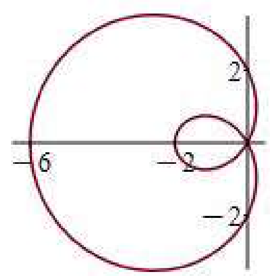
**a**. $(x-4)^2+(y-3)^2+(z-2)^2 = 4$
**b**. $(x-4)^2+(y+3)^2+(z-2)^2 = 25$
**c**. $(x-4)^2+(y-3)^2+(z-2)^2 = 25$ Correct
**d**. $(x+4)^2+(y+3)^2+(z+2)^2 = 4$
**e**. $(x+4)^2+(y+3)^2+(z+2)^2 = 25$

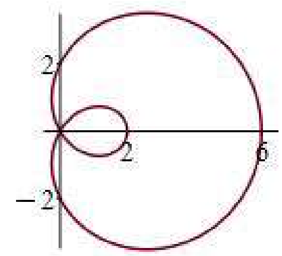
**Solution**: The equation of a sphere is $(x-a)^2+(y-b)^2+(z-c)^2 = R^2$ where the center is $(a,b,c) = (4,3,2)$ and the radius is the distance from the center to the $z$-axis which is $R = \sqrt{4^2+3^2} = 5$.

**2**. Which of the following is the plot of the polar curve $r = 4\cos\theta - 2$?

a. 


b. 

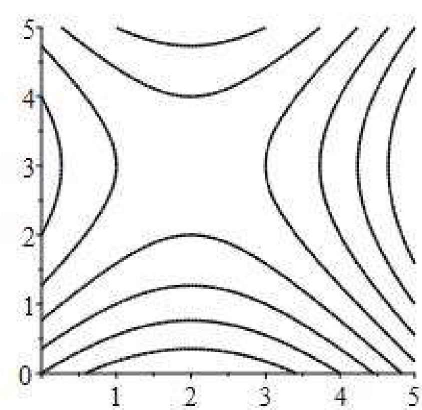

c. 


d. 


Correct

**Solution**: $r(0) = 4(1)-2 = 2$ (measured right since $\theta$ is right)
$r(\pi) = 4(-1)-2 = -6$ (measured right since $\theta$ is left and $r<0$).

**3**. The plot at the right is the contour plot of which function?




**a**. $z = (x-2)^2+(y-3)^2$
**b**. $z = (x-2)^2-(y-3)^2$ Correct
**c**. $z = (x-3)^2+(y-2)^2$
**d**. $z = (x-3)^2-(y-2)^2$

**Solution**: The plot is centered at $(2,3)$. So the function is $z = \pm(x-2)^2 \pm (y-3)^2$. Since the contours are hyperbolas, not circles, the function is $z = (x-2)^2-(y-3)^2$.

**4**. The force $\vec{F} = \langle 7, -3 \rangle$ pushes a mass from $P = (12, 1)$ to $Q = (7, -1)$.
Find the angle between the force and the displacement.

**a**. $135°$ Correct
**b**. $120°$
**c**. $60°$
**d**. $45°$
**e**. $30°$

**Solution**: The displacement is $\vec{D} = Q - P = \langle -5, -2 \rangle$. So the angle satisfies:

$$\cos\theta = \frac{\vec{F} \cdot \vec{D}}{|\vec{F}|\,|\vec{D}|} = \frac{-35 + 6}{\sqrt{25+4}\,\sqrt{49+9}} = \frac{-29}{\sqrt{29}\,\sqrt{58}} = \frac{-1}{\sqrt{2}} \qquad \theta = 135°$$

**5**. Do the vectors $\vec{u} = \langle 2, 0, 1 \rangle$, $\vec{v} = \langle 0, -1, 3 \rangle$ and $\vec{w} = \langle 3, 2, 0 \rangle$ form a left or right handed triplet?
Then find the volume of the parallelepiped with these edges.

**a**. left handed $V = 3$
**b**. left handed $V = 9$ Correct
**c**. left handed $V = -9$
**d**. right handed $V = 3$
**e**. right handed $V = -9$

**Solution**: $$\vec{u} \cdot \vec{v} \times \vec{w} = \begin{vmatrix} 2 & 0 & 1 \\ 0 & -1 & 3 \\ 3 & 2 & 0 \end{vmatrix} = 2\begin{vmatrix} -1 & 3 \\ 2 & 0 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = 2(-6) + 1(3) = -12 + 3 = -9$$

Since this is negative, the triplet is left handed. The volume is $V = |\vec{u} \cdot \vec{v} \times \vec{w}| = 9$

**6**. Find an equation of the plane through the point $P = (3, 2, 1)$
which is perpendicular to the line $(x, y, z) = (1 + 4t, 2 + 3t, 3 + 2t)$.
Then find where the plane passes through the $z$-axis.

**a**. $z = 2$
**b**. $z = 4$
**c**. $z = 5$
**d**. $z = 10$ Correct
**e**. $z = 20$

**Solution**: The normal to the plane is the direction of the line, $\vec{N} = \vec{v} = \langle 4, 3, 2 \rangle$. The point is given.
So the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $4x + 3y + 2z = 4 \cdot 3 + 3 \cdot 2 + 2 \cdot 1 = 20$.
The line intersects the $z$-axis when $x = y = 0$ or $2z = 20$. So $z = 10$

**7**. Classify the quadratic curve: $x^2 - 6x = 2y^2 - 4y - 7$.

**a**. parabola opening in the $x$ direction
**b**. parabola opening in the $y$ direction
**c**. hyperbola opening up and down
**d**. hyperbola opening left and right
**e**. cross    Correct

**Solution**: We take everything to one side of the equation and then complete the squares:

$$\begin{aligned} x^2 - 6x - 2y^2 + 4y &= -7 \\ (x^2 - 6x) - 2(y^2 - 2y) &= -7 \\ (x^2 - 6x + 9) - 2(y^2 - 2y + 1) &= -7 + 9 - 2 \\ (x-3)^2 - 2(y-1)^2 &= 0 \end{aligned}$$

Since the $x$ and $y$ terms are quadratic with opposite signs and the right side is $0$, the curve is a cross.

**8**. Your drone flies NorthEast $5\sqrt{2}$ km and then East $7$ km. If it flies home along a straight line, how far does it need to fly to get home?
**a**. $11$ km
**b**. $12$ km
**c**. $7 + 5\sqrt{2}$ km
**d**. $13$ km    Correct
**e**. $17$ km

**Solution**: The first vector is $\vec{u} = \langle 5, 5 \rangle$. The second vector is $\vec{v} = \langle 7, 0 \rangle$.
The flight home is the vector $\vec{w} = -\vec{u} - \vec{v} = \langle -12, 5 \rangle$. The distance home is $|\vec{w}| = \sqrt{12^2 + 5^2} = 13$ km.

**9**. Find the circulation in a bowl of water, counterclockwise around the circle $x^2 + y^2 = 16$, with $z = 3$, if its fluid velocity field is $\vec{V} = \langle x - y, x + y, 2z \rangle$.

**a**. $2\pi$
**b**. $4\pi$
**c**. $8\pi$
**d**. $16\pi$
**e**. $32\pi$    Correct

**Solution**: The circle may be parametrized by $\vec{r}(t) = (4\cos t, 4\sin t, 3)$.
Its tangent vector is $\vec{v} = \langle -4\sin t, 4\cos t, 0 \rangle$. The fluid velocity on the curve is
$\vec{V}(\vec{r}(t)) = \langle 4\cos t - 4\sin t, 4\cos t + 4\sin t, 6 \rangle$.
The dot product of the fluid velocity and the tangent vector is
$\vec{V}(\vec{r}(t)) \cdot \vec{v} = -4\sin t(4\cos t - 4\sin t) + 4\cos t(4\cos t + 4\sin t) + 0 = 16\sin^2 t + 16\cos^2 t = 16$
So the circulation is
Circ $= \oint \vec{V} \cdot d\vec{s} = \int_0^{2\pi} \vec{V}(\vec{r}(t)) \cdot \vec{v}\, dt = \int_0^{2\pi} 16\, dt = \left[ 16t \right]_0^{2\pi} = 32\pi$

Work Out: (Points indicated. Part credit possible. Show all work.)

**10**. (20 pts) Consider the twisted cubic $\vec{r} = (t^3, 3t^2, 6t)$. Compute each of the following.
Note: $t^4 + 4t^2 + 4 = (t^2 + 2)^2$

**a**. (6 pts) Arc length between $(0,0,0)$ and $(1,3,6)$.

**Solution**: $\vec{v} = \langle 3t^2, 6t, 6\rangle \qquad |\vec{v}| = \sqrt{9t^4 + 36t^2 + 36} = 3\sqrt{t^4 + 4t^2 + 4} = 3\sqrt{(t^2+2)^2} = 3(t^2+2)$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 3(t^2+2)\,dt = 3\left[\frac{t^3}{3} + 2t\right]_0^1 = 3\left[\frac{1}{3} + 2\right] = 7$$

**b**. (6 pts) Curvature $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$.
HINT: Factor out an $18^2$.

**Solution**: $\vec{a} = \langle 6t, 6, 0\rangle \qquad \vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2 & 6t & 6 \\ 6t & 6 & 0 \end{vmatrix} = \langle -36, 36t, 18t^2 - 36t^2\rangle = \langle -36, 36t, -18t^2\rangle$

$$|\vec{v} \times \vec{a}| = \sqrt{36^2 + 36^2t^2 + 18^2t^4} = 18\sqrt{4 + 4t^2 + t^4} = 18\sqrt{(t^2+2)^2} = 18(t^2+2)$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{18(t^2+2)}{3^3(t^2+2)^3} = \frac{2}{3(t^2+2)^2}$$

**c**. (4 pts) Tangential acceleration, $a_T$.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

**Solution**: $a_T = \dfrac{d}{dt}|\vec{v}| = \dfrac{d}{dt}3(t^2+2) = 6t$

**d**. (4 pts) Normal acceleration, $a_N$.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

**Solution**: $a_n = \kappa|\vec{v}|^2 = \dfrac{2}{3(t^2+2)^2}3^2(t^2+2)^2 = 6$

**11**. (10 pts) Find the average value of the function $f(x,y,z) = x^2$ on the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ for $0 \le t \le 2\pi$.

**Solution**: The velocity is $\vec{v} = \langle -3\sin t, 3\cos t, 4\rangle$. The speed is $|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$.
The length of the curve is $L = \int_0^{2\pi} |\vec{v}|\,dt = \int_0^{2\pi} 5\,dt = \Big[5t\Big]_0^{2\pi} = 10\pi$. The integral of $f$ is

$$\int f\,ds = \int_0^{2\pi} x^2|\vec{v}|\,dt = \int_0^{2\pi} 9\cos^2 t\,5\,dt = 45\int_0^{2\pi} \frac{1+\cos(2t)}{2}\,dt = \frac{45}{2}\left[t + \frac{\sin(2t)}{2}\right]_0^{2\pi} = \frac{45}{2}2\pi = 45\pi$$

So the average value is $f_{\text{ave}} = \frac{1}{L}\int f\,ds = \frac{1}{10\pi}45\pi = \frac{9}{2}$.

**12**. (10 pts) Write the vector $\vec{a} = \langle 5,5\rangle$ as the sum of vectors $\vec{p}$ and $\vec{q}$ where $\vec{p}$ is parallel to $\vec{b} = \langle 3,1\rangle$ and $\vec{q}$ is perpendicular to $\vec{b}$.

**Solution**:

$$\vec{p} = \text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b} = \frac{15+5}{9+1}\langle 3,1\rangle = 2\langle 3,1\rangle = \boxed{\langle 6,2\rangle} \qquad \vec{q} = \vec{a} - \vec{p} = \langle 5,5\rangle - \langle 6,2\rangle = \boxed{\langle -1,3\rangle}$$

Check: $\vec{p} + \vec{q} = \langle 6,2\rangle + \langle -1,3\rangle = \langle 5,5\rangle \qquad \vec{q}\cdot\vec{b} = \langle -1,3\rangle\cdot\langle 3,1\rangle = 0$

**13**. (10 pts) Consider the 2 planes:

$$P_1 : \quad 2x + y + 3z = 8$$
$$P_2 : \quad x + 2y - 2z = 7$$

Determine if they are parallel or intersecting. If they intersect, find a parametric equation for the line of intersection.
You MUST show why they are or are not parallel.

**Solution**: The normal vectors are $\vec{N}_1 = \langle 2,1,3\rangle$ and $\vec{N}_2 = \langle 1,2,-2\rangle$.
Since these are not proportional, the planes are not parallel.
The direction of the line of intersection is

$$\vec{v} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 3 \\ 1 & 2 & -2 \end{vmatrix} = \hat{\imath}(-8) - \hat{\jmath}(-7) + \hat{k}(3) = \langle -8,7,3\rangle$$

To find a point on the line of intersection, we pick $z = 0$ and solve

$$2x + y = 8$$
$$x + 2y = 7$$

The first equation minus twice the second gives: $y - 4y = 8 - 14$ or $-3y = -6$ or $y = 2$.
Then the second equation says $x = 7 - 2y = 7 - 2(2) = 3$. So a point is $P = (3,2,0)$.
So the line is $(x,y,z) = P + t\vec{v} = (3,2,0) + t\langle -8,7,3\rangle = (3 - 8t, 2 + 7t, 3t)$.