Name______________________________ Section:__________

MATH 221 Exam 1, Version C Fall 2023

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1-9	/54	12	/10
10	/20	13	/10
11	/10	Total	/104

Multiple Choice: (6 points each. No part credit.)

1. The circle $(x-2)^2 + (y-5)^2 = 9$ is tangent to which line?
HINT: Draw a picture.

a. $x = 1$

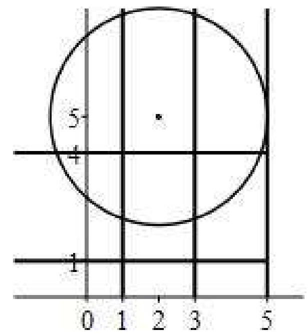
b. $x = 3$

c. $x = 5$ Correct

d. $y = 1$

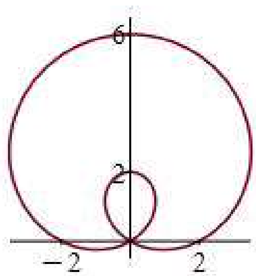
e. $y = 4$

Solution: See the plot.

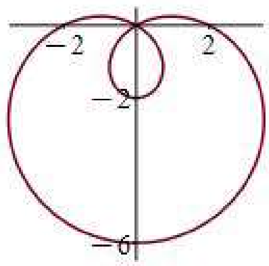


2. Which of the following is the plot of the polar curve $r = 4\cos\theta - 2$?

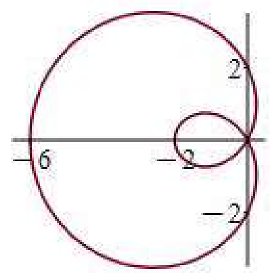
a.



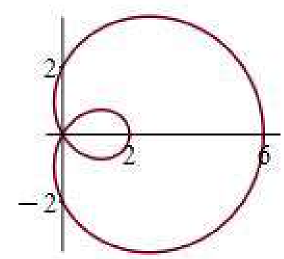
b.



c.



d.



Correct

Solution: $r(0) = 4(1) - 2 = 2$ (measured right since θ is right)

$r(\pi) = 4(-1) - 2 = -6$ (measured right since θ is left and $r < 0$).

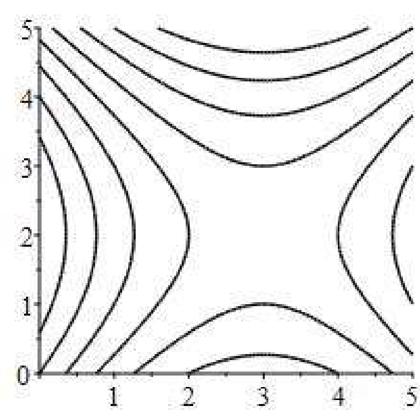
3. The plot at the right is the contour plot of which function?

a. $z = (x-2)^2 + (y-3)^2$

b. $z = (x-2)^2 - (y-3)^2$

c. $z = (x-3)^2 + (y-2)^2$

d. $z = (x-3)^2 - (y-2)^2$ Correct



Solution: The plot is centered at $(3,2)$. So the function is $z = \pm(x-3)^2 \pm (y-2)^2$.
Since the contours are hyperbolas, not circles, the function is $z = (x-3)^2 - (y-2)^2$.

4. At a certain point on a certain curve, $\vec{T} = \frac{1}{\sqrt{14}}\langle 3,2,-1\rangle$ and $\vec{B} = \frac{1}{\sqrt{21}}\langle 2,-1,4\rangle$. Find $\vec{N}$.

a. $\vec{N} = \frac{1}{\sqrt{6}}\langle -1,2,-1\rangle$
b. $\vec{N} = \frac{1}{\sqrt{6}}\langle -1,2,1\rangle$ Correct
c. $\vec{N} = \frac{1}{\sqrt{6}}\langle 1,2,-1\rangle$
d. $\vec{N} = \frac{1}{\sqrt{6}}\langle 1,-2,-1\rangle$
e. $\vec{N} = \frac{1}{\sqrt{6}}\langle -1,-2,1\rangle$

Solution: $\vec{N} = \vec{B} \times \vec{T} = \frac{1}{\sqrt{21}}\frac{1}{\sqrt{14}}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 4 \\ 3 & 2 & -1 \end{vmatrix}$

$$= \frac{1}{7\sqrt{6}}\left[\hat{\imath}(1-8) - \hat{\jmath}(-2-12) + \hat{k}(4+3)\right] = \frac{1}{\sqrt{6}}\langle -1,2,1\rangle$$

5. Find the area of the triangle with vertices $A - (1,1,1)$, $B = (5,1,3)$ and $C = (3,2,1)$.

a. 1
b. 3 Correct
c. 6
d. 12
e. 18

Solution: $\overrightarrow{AB} = B - A = \langle 4,0,2\rangle$ $\overrightarrow{AC} = C - A = \langle 2,1,0\rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix} = \hat{\imath}(-2) - \hat{\jmath}(-4) + \hat{k}(4) = \langle -2,4,4\rangle$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}|\langle -2,4,4\rangle| = \frac{1}{2}\sqrt{4+16+16} = \frac{1}{2}\sqrt{36} = 3$$

6. Find an equation of the line through the point $P = (2,3,4)$
which is perpendicular to the plane $4x + 3y + 2z = 15$.
Then find the point where the line passes through the xy-plane.

a. $(x,y,z) = (10,9,0)$
b. $(x,y,z) = (-10,-9,0)$
c. $(x,y,z) = (-6,-3,0)$ Correct
d. $(x,y,z) = (6,3,0)$
e. $(x,y,z) = (6,6,0)$

Solution: The direction of the line is the normal to the plane, $\vec{v} = \vec{N} = \langle 4,3,2\rangle$. The point is given.
So the line is $X = P + t\vec{v}$ or $(x,y,z) = (2,3,4) + t\langle 4,3,2\rangle$ or $x = 2 + 4t$ $y = 3 + 3t$ $z = 4 + 2t$.
The line intersects the xy-plane when $z = 0 = 4 + 2t$ or $t = -2$. So
$(x,y,z) = (2,3,4) - 2\langle 4,3,2\rangle = (-6,-3,0)$

7. Classify the quadratic surface: $2x^2 - 8x + y^2 - 6y - z^2 + 2z = -17$.

a. elliptic paraboloid opening up in the z direction
b. elliptic paraboloid opening down in the z direction
c. hyperboloid of 1-sheet
d. hyperboloid of 2-sheets   Correct
e. cone

Solution: We complete the squares:

$$\begin{aligned} 2x^2 - 8x + y^2 - 6y - z^2 + 2z &= -17 \\ 2(x^2 - 4x) + (y^2 - 6y) - (z^2 - 2z) &= -17 \\ 2(x^2 - 4x + 4) + (y^2 - 6y + 9) - (z^2 - 2z + 1) &= -17 + 8 + 9 - 1 \\ 2(x-2)^2 + (y-3)^2 - (z-1)^2 &= -1 \\ -2(x-2)^2 - (y-3)^2 + (z-1)^2 &= 1 \end{aligned}$$

Since the x, y and z terms are quadratic, the right side is 1 and there are 2 minuses and 1 plus, the surface is a hyperboloid of 2-sheets.

8. If an airplane is flying from East to West directly above the equator, where does $\vec{B}$ point? Why?

a. North
b. South   Correct
c. West
d. Up
e. Down

Solution: $\vec{T}$ points West. $\vec{N}$ points Down toward the center of the Earth.
So $\vec{B} = \vec{T} \times \vec{N}$ points South.

9. Find the work done by the force $\vec{F} = \langle z^2, yz, xz \rangle$ to move a bead along the twisted cubic $\vec{r}(t) = (t, t^2, t^3)$ from $t = 0$ to $t = 1$.

a. 1
b. $\frac{2}{7}$
c. $\frac{3}{7}$
d. $\frac{6}{7}$ Correct
e. $\frac{12}{7}$

Solution: The velocity is $\vec{v} = \langle 1, 2t, 3t^2 \rangle$. The force along the curve is $\vec{F}(\vec{r}(t)) = \langle t^6, t^5, t^4 \rangle$.
So the work done is

$$W = \int_0^1 \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(\vec{r}(t)).\vec{v}\,dt = \int_0^1 (t^6 + 2t^6 + 3t^6)\,dt = 6\int_0^1 t^6\,dt = \left[6\frac{t^7}{7}\right]_0^1 = \frac{6}{7}.$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (20 pts) Consider the twisted cubic $\vec{r} = (3t, 3t^2, 2t^3)$. Compute each of the following.
Note: $1 + 4t^2 + 4t^4 = (1 + 2t^2)^2$

a. (6 pts) Arc length between $(0,0,0)$ and $(3,3,2)$.

Solution: $\vec{v} = \langle 3, 6t, 6t^2 \rangle$ $\quad |\vec{v}| = \sqrt{9 + 36t^2 + 36t^4} = 3\sqrt{1 + 4t^2 + 4t^4} = 3\sqrt{(1 + 2t^2)^2} = 3(1 + 2t^2)$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 3(1 + 2t^2)\,dt = 3\left[t + \frac{2t^3}{3}\right]_0^1 = 3\left[1 + \frac{2}{3}\right] = 5$$

b. (6 pts) Curvature $\kappa = \dfrac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$.
HINT: Factor out an 18^2.

Solution: $\vec{a} = \langle 0, 6, 12t \rangle$ $\quad \vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 6t & 6t^2 \\ 0 & 6 & 12t \end{vmatrix} = \langle 72t^2 - 36t^2, -36t, 18 \rangle = \langle 36t^2, -36t, 18 \rangle$

$$|\vec{v} \times \vec{a}| = \sqrt{36^2 t^4 + 36^2 t^2 + 18^2} = 18\sqrt{4t^4 + 4t^2 + 1} = 18\sqrt{(1 + 2t^2)^2} = 18(1 + 2t^2)$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{18(1 + 2t^2)}{3^3(1 + 2t^2)^3} = \frac{2}{3(1 + 2t^2)^2}$$

c. (4 pts) Tangential acceleration, a_T.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

Solution: $a_T = \dfrac{d}{dt}|\vec{v}| = \dfrac{d}{dt}3(1 + 2t^2) = 12t$

d. (4 pts) Normal acceleration, a_N.
HINT: You do NOT need to compute $\hat{T}$, $\hat{N}$ or $\hat{B}$.

Solution: $a_n = \kappa|\vec{v}|^2 = \dfrac{2}{3(1 + 2t^2)^2} 3^2(1 + 2t^2)^2 = 6$

11. (10 pts) Find the average value of the function $f(x,y,z) = y^2$ on the helix $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ for $0 \le t \le 2\pi$.

Solution: The velocity is $\vec{v} = \langle -3\sin t, 3\cos t, 4\rangle$. The speed is $|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$.
The length of the curve is $L = \int_0^{2\pi} |\vec{v}|\,dt = \int_0^{2\pi} 5\,dt = \Big[5t\Big]_0^{2\pi} = 10\pi$. The integral of f is

$$\int f\,ds = \int_0^{2\pi} y^2 |\vec{v}|\,dt = \int_0^{2\pi} 9\sin^2 t\, 5\,dt = 45\int_0^{2\pi} \frac{1-\cos(2t)}{2}\,dt = \frac{45}{2}\left[t - \frac{\sin(2t)}{2}\right]_0^{2\pi} = \frac{45}{2}2\pi = 45\pi$$

So the average value is $f_{\text{ave}} = \frac{1}{L}\int f\,ds = \frac{1}{10\pi}45\pi = \frac{9}{2}$.

12. (10 pts) Write the vector, $\langle 1,1,3\rangle$, as a linear composition of $\langle 2,1,3\rangle$ and $\langle 3,1,2\rangle$, i.e. find a and b so that:

$$\langle 1,1,3\rangle = a\langle 2,1,3\rangle + b\langle 3,1,2\rangle$$

or show it cannot be done.

Solution: We need to solve the equations: $2a + 3b = 1 \quad a + b = 1 \quad 3a + 2b = 3$
The first equation minus twice the second is $\boxed{b = -1}$. Then the second equation says $\boxed{a = 2}$.
We check the third equation: $3a + 2b = 3(2) + 2(-1) = 4 \ne 3$. So $\boxed{\text{there is no solution}}$.

13. (10 pts) Consider the 2 lines:

$$L_1(t): \quad (x,y,z) = (7+2t, 4+t, 3+t)$$
$$L_2(t): \quad (x,y,z) = (5-2t, 1+t, -2+3t)$$

Determine if they are parallel, intersecting or skew. If they intersect, find the point of intersection.
You MUST show why they are or are not parallel or skew.

Solution: The direction vectors are $\vec{v}_1 = \langle 2,1,1\rangle$ and $\vec{v}_2 = \langle -2,1,3\rangle$.
Since these are not proportional, the lines are not parallel.
We equate the x, y and z components of the lines and change the second parameter to s:

$$7 + 2t = 5 - 2s$$
$$4 + t = 1 + s$$
$$3 + t = -2 + 3s$$

The second equation says $s = 3 + t$. We substitute into the first equation and solve for t and s:

$$7 + 2t = 5 - 2(3+t) = -1 - 2t \quad \Rightarrow \quad 4t = -8 \quad t = -2 \quad s = 1$$

The points on the lines with these parameter vaues are:

$$L_1(-2): \quad (x,y,z) = (7 + 2(-2), 4 + (-2), 3 + (-2)) = (3,2,1)$$
$$L_2(1): \quad (x,y,z) = (5 - 2(1), 1 + (1), -2 + 3(1)) = (3,2,1)$$

Since these are equal, they are not skew and they intersect at $(x,y,z) = (3,2,1)$.