Name________________________________ Section:__________

MATH 221 Exam 2, Version A Fall 2023

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1-8	/48	10	/20
9	/36	Total	/104

Multiple Choice: (6 points each. No part credit.)

1. Consider the function $z = f(x,y) = xy^3$. Its y-trace with $x = 2$ is the intersection of the graph of $z = f(x,y)$ and the plane $x = 2$. Find the slope of this y-trace at $y = 3$.

a. 8

b. 24

c. 27

d. 54 Correct

e. 96

Solution: The slope of the y-trace is the y-partial derivative Here it is $f_y(x,y) = 3xy^2$. So the slope of the y-trace with $x = 2$ at $y = 3$ is $f_y(2,3) = 3 \cdot 2 \cdot 3^2 = 54$.

2. Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^4+y^2}$.

Which of the following paths of approach gives a different value of the limit?

a. $y = 2x$ & $x \to 0$

b. $y = x^2 + x^3$ & $x \to 0$

c. $y = x^2 + x$ & $x \to 0$

d. $y = x^2$ & $x \to 0$

e. They are all equal. Correct

Hint: Don't bother multiplying out any quadratic.

Solution:

$$\lim_{\substack{y=2x\\x\to0}} \frac{xy^2}{x^4+y^2} = \lim_{x\to0} \frac{x4x^2}{x^4+4x^2} = \lim_{x\to0} \frac{4x}{x^2+4} = 0$$

$$\lim_{\substack{y=x^2+x^3\\x\to0}} \frac{xy^2}{x^4+y^2} = \lim_{x\to0} \frac{x(x^2+x^3)^2}{x^4+(x^2+x^3)^2} = \lim_{x\to0} \frac{x(1+x)^2}{1+(1+x)^2} = 0$$

$$\lim_{\substack{y=x^2+x\\x\to0}} \frac{xy^2}{x^4+y^2} = \lim_{x\to0} \frac{x(x^2+x)^2}{x^4+(x^2+x)^2} = \lim_{x\to0} \frac{x(x+1)^2}{x^2+(x+1)^2} = 0$$

$$\lim_{\substack{y=x^3\\x\to0}} \frac{xy^2}{x^4+y^2} = \lim_{x\to0} \frac{x(x^3)^2}{x^4+(x^3)^2} = \lim_{x\to0} \frac{x^3}{1+x^2} = 0$$

3. Find the plane tangent to the graph of the function $z = x^2 e^{(2y-4)}$ at the point $(3,2)$. Its z-intercept is

a. $c = -45$ Correct

b. $c = -33$

c. $c = -9$

d. $c = 9$

e. $c = 33$

Solution: $f(x,y) = x^2 e^{(2y-4)}$ $f(3,2) = 9$

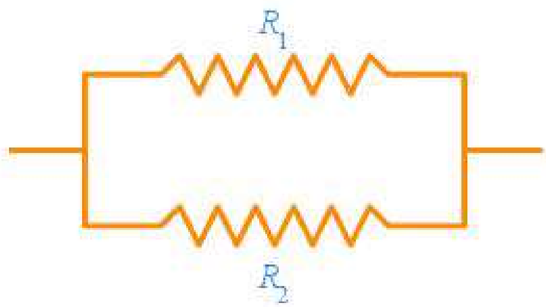
$f_x(x,y) = 2xe^{(2y-4)}$ $f_x(3,2) = 6$ $f_y(x,y) = x^2 2e^{(2y-4)}$ $f_y(3,2) = 18$

$$z = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) = 9 + 6(x-3) + 18(y-2)$$
$$= 6x + 18y + 9 - 18 - 36 = 6x + 18y - 45 \qquad c = -45$$

4. If two resistors, with resistances R_1 and R_2 are connected in parallel, then the total resistance R satisfies:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R = \frac{R_1 R_2}{R_1 + R_2}$$

Currently $R_1 = 200$ ohms and $R_2 = 400$ ohms. If they increase by $\Delta R_1 = 1.8$ ohms and $\Delta R_2 = 0.9$ ohms, use the linear approximation to approximate how much R changes.



a. $\Delta R \approx 0.1$ ohms

b. $\Delta R \approx 0.3$ ohms

c. $\Delta R \approx 0.6$ ohms

d. $\Delta R \approx 0.9$ ohms Correct

e. $\Delta R \approx 1.8$ ohms

Solution: The derivatives of R are

$$\frac{\partial R}{\partial R_1} = \frac{(R_1 + R_2)R_2 - R_1 R_2(1)}{(R_1 + R_2)^2} = \frac{R_2{}^2}{(R_1 + R_2)^2} = \frac{400^2}{600^2} = \frac{4}{9} \qquad \frac{\partial R}{\partial R_2} = \frac{R_1{}^2}{(R_1 + R_2)^2} = \frac{200^2}{600^2} = \frac{1}{9}$$

The change in R is approximately its differential:

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 = \frac{4}{9} 1.8 + \frac{1}{9} 0.9 = 0.9 \text{ ohms}$$

5. Let $f(x,y) = x\sin(xy)$. Compute $f_{xy}\left(2, \frac{\pi}{8}\right)$.

a. $2\sqrt{2} + \frac{1}{4}\sqrt{2}\,\pi$

b. $2\sqrt{2} - \frac{1}{4}\sqrt{2}\,\pi$ Correct

c. $-2\sqrt{2} + \frac{1}{4}\sqrt{2}\,\pi$

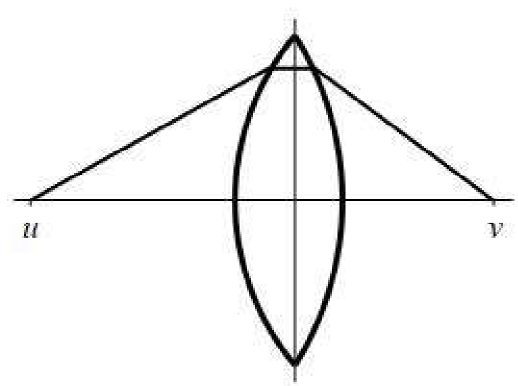
d. $-2\sqrt{2} - \frac{1}{4}\sqrt{2}\,\pi$

Solution: It's easier to compute f_{yx}. $f_y = x^2\cos(xy)$ $f_{yx} = 2x\cos(xy) - x^2 y\sin(xy)$

$f_{yx}\left(2, \frac{\pi}{8}\right) = 4\cos\left(\frac{\pi}{4}\right) - 4\frac{\pi}{8}\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2} - \frac{1}{4}\sqrt{2}\,\pi$

6. If the focal length of a lens is f and an object is placed a distance u from the lens, then the image of the object will appear to be a distance v from the lens related by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \quad \text{or} \quad v = \frac{fu}{u-f}$$



Consider a lens with an adjustable focal length.

Currently, $f = 2$ cm, $u = 8$ cm and $v = \frac{8}{3}$ cm.

If f and u are increasing at $\frac{df}{dt} = 0.9\ \frac{\text{cm}}{\text{sec}}$ and $\frac{du}{dt} = 1.8\ \frac{\text{cm}}{\text{sec}}$, at what rate is v changing?

a. $\frac{dv}{dt} = 1\ \frac{\text{cm}}{\text{sec}}$

b. $\frac{dv}{dt} = 1.2\ \frac{\text{cm}}{\text{sec}}$

c. $\frac{dv}{dt} = 1.4\ \frac{\text{cm}}{\text{sec}}$ Correct

d. $\frac{dv}{dt} = 1.6\ \frac{\text{cm}}{\text{sec}}$

e. $\frac{dv}{dt} = 1.8\ \frac{\text{cm}}{\text{sec}}$

Solution: We take the time derivative of the relation:

$$-\frac{1}{f^2}\frac{df}{dt} = -\frac{1}{u^2}\frac{du}{dt} - \frac{1}{v^2}\frac{dv}{dt}$$

and solve for $\frac{dv}{dt}$

$$\frac{dv}{dt} = \frac{v^2}{f^2}\frac{df}{dt} - \frac{v^2}{u^2}\frac{du}{dt} = \left(\frac{8}{3}\right)^2\frac{1}{2^2}0.9 - \left(\frac{8}{3}\right)^2\frac{1}{8^2}1.8 = \left(\frac{4}{3}\right)^2 0.9 - \left(\frac{1}{3}\right)^2 1.8$$

$$16 \cdot 0.1 - 0.2 = 1.4\ \frac{\text{cm}}{\text{sec}}$$

7. Find the equation of the plane tangent to $x^2z^2 - yz = 3$ at the point $P = (1,2,3)$.
It's z-intercept is:

a. $c = 3$

b. $c = 4$

c. $c = 6$ Correct

d. $c = 12$

e. $c = 24$

Solution: Let $F(x,y,x) = x^2z^2 - yz$. Then $\vec{\nabla}F = \langle 2xz^2, -z, 2x^2z - y\rangle$ and
$\vec{N} = \vec{\nabla}F\big|_P = \langle 18,-3,4\rangle$. So the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or
$18x - 3y + 4z = 18 \cdot 1 - 3 \cdot 2 + 4 \cdot 3 = 24$
The z-intercept occurs when $x = y = 0$. So the z-intercept is $c = \frac{24}{4} = 6$.

8. The point $(x,y) = (1,1)$ is a critical point of the function $f(x,y) = 3x^2 - 6xy - 2y^3 + 6y^2$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum

b. Local Maximum

c. Inflection Point

d. Saddle Point Correct

e. Test Fails

Solution: $f_x = 6x - 6y$ $f_y = -6x - 6y^2 + 12y$
We check: $f_x(1,1) = 6 - 6 = 0$ $f_y(1,1) = -6 - 6 + 12 = 0$, Yes, it's a critical point.
$f_{xx} = 6$ $f_{yy} = -12y$ $f_{xy} = -6$
$f_{xx}(1,1) = 6$ $f_{yy}(1,1) = -12$ $f_{xy}(1,1) = -6$
$D = f_{xx}f_{yy} - f_{xy}{}^2 = (6)(-12) - (-6)^2 = -72 - 36 < 0$ SaddlePoint

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (36 pts) Obi-Two is flying the Centurion Eagle through the Force, Desperation and Luck fields. The Force, F, depends on the Desperation, D, and Luck, L, by the relation: $F = D^3 L$. Currently, the Desperation and Luck and their gradients are:

$$D = 2 \qquad \vec{\nabla}D = \langle 0,1,2 \rangle$$
$$L = 3 \qquad \vec{\nabla}L = \langle 3,1,0 \rangle$$

a. (23 pts) Obi-Two's current velocity is $\vec{v} = \langle 2,1,3 \rangle$. Find the rate that Obi-Two sees the Force changing.

Solution: The partial derivatives of $\vec{F}$ are:

$$\frac{\partial F}{\partial D} = 3D^2 L = 3 \cdot 2^2 \cdot 3 = 36 \qquad \frac{\partial F}{\partial L} = D^3 = 2^2 = 8$$

The derivatives of x, y, z, D and L are components of the velocity and the gradients. So by the Chain Rule, the derivative of F is:

$$\frac{dF}{dt} = \frac{\partial F}{\partial D}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) + \frac{\partial F}{\partial L}\left(\frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} + \frac{\partial L}{\partial z}\frac{dz}{dt}\right)$$
$$= 36(1 \cdot 1 + 2 \cdot 3) + 8(3 \cdot 2 + 1 \cdot 1) = 252 + 56 = 308$$

b. (13 pts) In what direction should Obi-Two travel to increase the Force as fast as possible? HINT: Compute each (x,y,z) partial derivative separately.

Solution: The partial derivatives of $\vec{F}$ are:

$$\frac{\partial F}{\partial D} = 3D^2 L = 3 \cdot 2^2 \cdot 3 = 36 \qquad \frac{\partial F}{\partial L} = D^3 = 2^2 = 8$$

The components of the gradient are the (x,y,z) partial derivatives of $\vec{F}$:

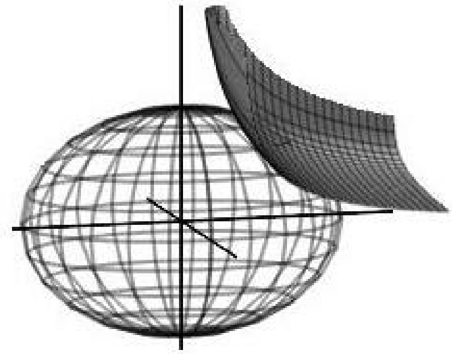
$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial x} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial x} = 36 \cdot 0 + 8 \cdot 3 = 24$$
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial y} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial y} = 36 \cdot 1 + 8 \cdot 1 = 44$$
$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial z} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial z} = 36 \cdot 2 + 8 \cdot 0 = 72$$

The Force increases fastest in the direction $\vec{\nabla}F = \langle 24,44,72 \rangle$.
Alternatively:

$$\vec{\nabla}F = \frac{\partial F}{\partial D}\vec{\nabla}D + \frac{\partial F}{\partial L}\vec{\nabla}L$$

10. (20 pts) Find the largest value of the function $f = xy^2z^4$ on the ellipsoid:

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{2} = \frac{7}{2}$$



Solution: We maximize $f = xy^2z^4$ with the constraint $g = \frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{2} = \frac{7}{2}$. The gradients are:

$\vec{\nabla}f = \langle y^2z^4, 2xyz^4, 4xy^2z^3 \rangle$ $\quad\quad$ $\vec{\nabla}g = \left\langle x, \frac{y}{2}, z \right\rangle$ $\quad$ The Lagrange equations are;

$y^2z^4 = \lambda x \quad\quad 2xyz^4 = \lambda \frac{y}{2} \quad\quad 4xy^2z^3 = \lambda z$

Multiply the first by x, the second by $\frac{y}{2}$ and the third by $\frac{z}{4}$ and equate:

$xy^2z^4 = \lambda x^2 = \lambda \frac{y^2}{4} = \lambda \frac{z^2}{4}$ $\quad\quad$ So $x^2 = \frac{z^2}{4} \quad\quad y^2 = z^2$ $\quad\quad$ Plug into the constraint:

$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{2} = \frac{z^2}{8} + \frac{z^2}{4} + \frac{z^2}{2} = \frac{7}{8}z^2 = \frac{7}{2} \quad\quad z^2 = 4$

$z = 2 \quad\quad x = \frac{1}{2}z = 1 \quad\quad y = z = 2 \quad\quad f = 1 \cdot 2^2 \cdot 2^4 = 2^6 = 64$