Name________________________________ Section:__________

MATH 221 Exam 2, Version B Fall 2023

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1-8	/48	10	/20
9	/36	Total	/104

Multiple Choice: (6 points each. No part credit.)

1. Consider the function $z = f(x,y) = xy^3$. Its x-trace with $y = 3$ is the intersection of the graph of $z = f(x,y)$ and the plane $y = 3$. Find the slope of this x-trace at $x = 2$.

a. 8

b. 24

c. 27 Correct

d. 54

e. 96

Solution: The slope of the x-trace is the x-partial derivative Here it is $f_x(x,y) = y^3$. So the slope of the x-trace with $y = 3$ at $x = 2$ is $f_x(2,3) = 3^3 = 27$.

2. Consider the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4 + y^2}$.

Which of the following paths of approach gives a different value of the limit?

a. $y = 2x$ & $x \to 0$

b. $y = x^2 + x^3$ & $x \to 0$ Correct

c. $y = x^2 + x$ & $x \to 0$

d. $y = x^3$ & $x \to 0$

e. They are all equal.

Hint: Don't bother multiplying out any quadratic.

Solution:

$$\lim_{\substack{y=2x\\x\to0}} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2 2x}{x^4+4x^2} = \lim_{x\to0} \frac{2x}{x^2+4} = 0$$

$$\lim_{\substack{y=x^2+x^3\\x\to0}} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2(x^2+x^3)}{x^4+(x^2+x^3)^2} = \lim_{x\to0} \frac{(1+x)}{1+(1+x)^2} = \frac{1}{2}$$

$$\lim_{\substack{y=x^2+x\\x\to0}} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2(x^2+x)}{x^4+(x^2+x)^2} = \lim_{x\to0} \frac{x^2+x}{x^2+(x+1)^2} = 0$$

$$\lim_{\substack{y=x^3\\x\to0}} \frac{x^2y}{x^4+y^2} = \lim_{x\to0} \frac{x^2(x^3)}{x^4+(x^3)^2} = \lim_{x\to0} \frac{x}{1+x} = 0$$

3. Find the plane tangent to the graph of the function $z = x\cos y$ at the point $\left(\sqrt{2}, \frac{\pi}{4}\right)$.
Its z-intercept is

a. $c = \frac{\pi}{4}$ Correct

b. $c = -\frac{\pi}{4}$

c. $c = \frac{\pi}{4}\sqrt{2}$

d. $c = -\frac{\pi}{2}\sqrt{2}$

e. $c = 1 + \sqrt{2} - \frac{\pi}{4}$

Solution: $f(x,y) = x\cos y \qquad f\left(\sqrt{2}, \frac{\pi}{4}\right) = \sqrt{2}\,\frac{1}{\sqrt{2}} = 1$

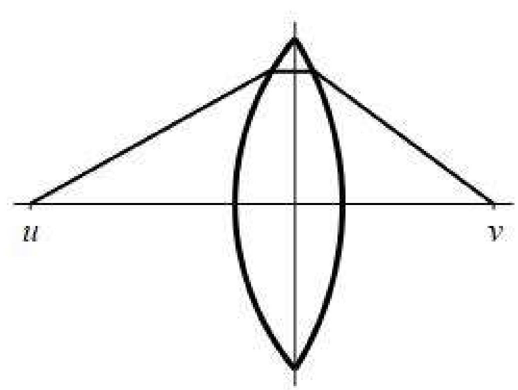
$f_x(x,y) = \cos y \qquad f_x\left(\sqrt{2}, \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \qquad f_y(x,y) = -x\sin y \qquad f_y\left(\sqrt{2}, \frac{\pi}{4}\right) = -\sqrt{2}\,\frac{1}{\sqrt{2}} = -1$

$$z = f\left(\sqrt{2}, \frac{\pi}{4}\right) + f_x\left(\sqrt{2}, \frac{\pi}{4}\right)\left(x - \sqrt{2}\right) + f_y\left(\sqrt{2}, \frac{\pi}{4}\right)\left(y - \frac{\pi}{4}\right) = 1 + \frac{1}{\sqrt{2}}\left(x - \sqrt{2}\right) - 1\left(y - \frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}x - y + 1 - 1 + \frac{\pi}{4} = 12x - 16y + \frac{\pi}{4} \qquad c = \frac{\pi}{4}$$

4. If the focal length of a lens is f and an object is placed a distance u from the lens, then the image of the object will appear to be a distance v from the lens related by

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \quad \text{or} \quad v = \frac{fu}{u-f}$$



Consider a lens with an adjustable focal length.
Currently, $f = 2$ cm, $u = 8$ cm and $v = \frac{8}{3}$ cm.
If f and u increase by $\Delta f = 0.9$ cm and $\Delta u = 1.8$ cm, use the linear approximation to approximate how much v changes.

a. $\Delta v \approx 1$ cm

b. $\Delta v \approx 1.2$ cm

c. $\Delta v \approx 1.4$ cm Correct

d. $\Delta v \approx 1.6$ cm

e. $\Delta v \approx 1.8$ cm

Solution: The derivatives of v are

$$\frac{\partial v}{\partial f} = \frac{(u-f)u - fu(-1)}{(u-f)^2} = \frac{u^2}{(u-f)^2} = \frac{8^2}{6^2} = \frac{16}{9} \qquad \frac{\partial v}{\partial u} = \frac{(u-f)f - fu(1)}{(u-f)^2} = \frac{-2^2}{6^2} = \frac{-1}{9}$$

The change in v is approximately its differential:

$$\Delta v \approx dv = \frac{\partial v}{\partial f}df + \frac{\partial v}{\partial u}du = \frac{16}{9}0.9 + \frac{-1}{9}1.8 = 1.4 \text{ cm}$$

5. Let $f(x,y) = x\cos(xy)$. Compute $f_{xy}\left(2, \frac{\pi}{8}\right)$.

a. $2\sqrt{2} + \frac{1}{4}\sqrt{2}\,\pi$

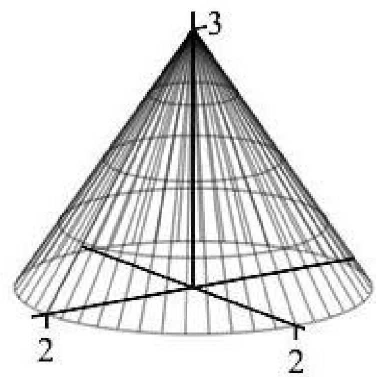
b. $2\sqrt{2} - \frac{1}{4}\sqrt{2}\,\pi$

c. $-2\sqrt{2} + \frac{1}{4}\sqrt{2}\,\pi$

d. $-2\sqrt{2} - \frac{1}{4}\sqrt{2}\,\pi$ Correct

Solution: It's easier to compute f_{yx}. $f_y = -x^2\sin(xy)$ $f_{yx} = -2x\sin(xy) - x^2y\cos(xy)$
$f_{yx}\left(2, \frac{\pi}{8}\right) = -4\sin\left(2 \cdot \frac{\pi}{8}\right) - 4\frac{\pi}{8}\cos\left(2 \cdot \frac{\pi}{8}\right) = -2\sqrt{2} - \frac{1}{4}\sqrt{2}\,\pi$

6. The volume of a cone is $V = \frac{1}{12}\pi D^2 H$ where D is the diameter and H is the height.
Currently, $D = 4$ cm, $H = 3$ cm and $V = 4\pi$ cm.
If V and H are increasing at $\frac{dV}{dt} = 0.6\pi\ \frac{\text{cm}}{\text{sec}}$ and $\frac{dH}{dt} = 0.3\ \frac{\text{cm}}{\text{sec}}$, at what rate is D changing?



a. $\frac{dD}{dt} = 0.1\ \frac{\text{cm}}{\text{sec}}$ Correct

b. $\frac{dD}{dt} = 0.25\ \frac{\text{cm}}{\text{sec}}$

c. $\frac{dD}{dt} = 0.5\ \frac{\text{cm}}{\text{sec}}$

d. $\frac{dD}{dt} = 0.6\ \frac{\text{cm}}{\text{sec}}$

e. $\frac{dD}{dt} = 0.75\ \frac{\text{cm}}{\text{sec}}$

Solution: We take the time derivative of the volume:

$$\frac{dV}{dt} = \frac{\partial V}{\partial D}\frac{dD}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt} = \frac{1}{6}\pi DH\frac{dD}{dt} + \frac{1}{12}\pi D^2\frac{dH}{dt}$$

plug in numbers $0.6\pi = 2\pi\frac{dD}{dt} + \frac{4}{3}\pi 0.3$ and solve for $\frac{dD}{dt} = \frac{1}{2}(0.6 - 0.4) = 0.1$

7. Find the equation of the plane tangent to $x^2y^2e^{2z-4} = 1$ at the point $P = (1,1,2)$.
It's z-intercept is:

a. $c = 0$

b. $c = 1$

c. $c = 2$

d. $c = 3$

e. $c = 4$ Correct

Solution: Let $F(x,y,x) = x^2ye^{z^2-4}$. Then $\vec{\nabla}F = \langle 2xy^2e^{2z-4}, 2x^2ye^{2z-4}, x^2y^22e^{2z-4}\rangle$ and $\vec{N} = \vec{\nabla}F\Big|_P = \langle 2,2,2\rangle$. So the plane is $\vec{N}\cdot X = \vec{N}\cdot P$ or
$2x + 2y + 2z = 2\cdot 1 + 2\cdot 1 + 2\cdot 2 = 8$ or $x + y + z = 4$
The z-intercept occurs when $x = y = 0$. So the z-intercept is $c = 4$.

8. The point $(x,y) = (1,2)$ is a critical point of the function $f(x,y) = 2x^3y^2 + x^2y^3 - 5x^2y^2$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum Correct

b. Local Maximum

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $f_x = 6x^2y^2 + 2xy^3 - 10xy^2$ $f_y = 4x^3y + 3x^2y^2 - 10x^2y$
We check: $f_x(1,2) = 24 + 16 - 40 = 0$ $f_y(1,2) = 8 + 12 - 20 = 0$, Yes, it's a critical point.
$f_{xx} = 12xy^2 + 2y^3 - 10y^2$ $f_{yy} = 4x^3 + 6x^2y - 10x^2$ $f_{xy} = 12x^2y + 6xy^2 - 20xy$
$f_{xx}(1,2) = 48 + 16 - 40 = 24 > 0$ $f_{yy}(1,2) = 4 + 12 - 10 = 6 > 0$ $f_{xy}(1,2) = 24 + 24 - 40 = 8$
$D = f_{xx}f_{yy} - {f_{xy}}^2 = 24\cdot 6 - 8^2 = 80 > 0$ Local Minimum

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (36 pts) Obi-Two is flying the Centurion Eagle through the Force, Desperation and Luck fields. The Force, F, depends on the Desperation, D, and Luck, L, by the relation: $F = 2D^2L$. Currently, the Desperation and Luck and their gradients are:

$$D = 2 \qquad \vec{\nabla}D = \langle 1,0,3 \rangle$$
$$L = 3 \qquad \vec{\nabla}L = \langle 2,1,0 \rangle$$

a. (23 pts) Obi-Two's current velocity is $\vec{v} = \langle 3,2,1 \rangle$. Find the rate that Obi-Two sees the Force changing.

Solution: The partial derivatives of $\vec{F}$ are:

$$\frac{\partial F}{\partial D} = 4DL = 4 \cdot 2 \cdot 3 = 24 \qquad \frac{\partial F}{\partial L} = 2D^2 = 2 \cdot 2^2 = 8$$

The derivatives of x, y, z, D and L are components of the velocity and the gradients. So by the Chain Rule, the derivative of F is:

$$\frac{dF}{dt} = \frac{\partial F}{\partial D}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) + \frac{\partial F}{\partial L}\left(\frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} + \frac{\partial L}{\partial z}\frac{dz}{dt}\right)$$
$$= 24(1 \cdot 3 + 3 \cdot 1) + 8(2 \cdot 3 + 1 \cdot 2) = 144 + 64 = 208$$

b. (13 pts) In what direction should Obi-Two travel to increase the Force as fast as possible? HINT: Compute each (x,y,z) partial derivative separately.

Solution: The partial derivatives of $\vec{F}$ are:

$$\frac{\partial F}{\partial D} = 4DL = 4 \cdot 2 \cdot 3 = 24 \qquad \frac{\partial F}{\partial L} = 2D^2 = 2 \cdot 2^2 = 8$$

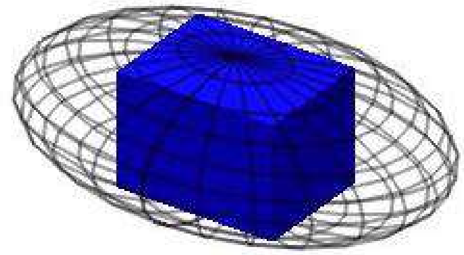
The components of the gradient are the (x,y,z) partial derivatives of $\vec{F}$:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial x} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial x} = 24 \cdot 1 + 8 \cdot 2 = 40$$
$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial y} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial y} = 24 \cdot 0 + 8 \cdot 1 = 8$$
$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial z} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial z} = 24 \cdot 3 + 8 \cdot 0 = 72$$

The Force increases fastest in the direction $\vec{\nabla}F = \langle 40,8,72 \rangle$.
Alternatively:

$$\vec{\nabla}F = \frac{\partial F}{\partial D}\vec{\nabla}D + \frac{\partial F}{\partial L}\vec{\nabla}L$$

10. (20 pts) Find the volume of the largest rectangular solid with its 8 vertices on the ellipsoid:



$$\frac{x^2}{8} + \frac{y^2}{18} + \frac{z^2}{2} = 6$$

Solution: Let (x,y,z) be the vertex in the first octant.
The length is $2x$. The width is $2y$. The height is $2z$. We maximize the volume $V = 8xyz$.
The constraint is $g = \frac{x^2}{8} + \frac{y^2}{18} + \frac{z^2}{2} = 6$. Their gradients are:
$\vec{\nabla} V = \langle 8yz, 8xz, 8xy \rangle$ $\vec{\nabla} g = \left\langle \frac{x}{4}, \frac{y}{9}, z \right\rangle$ The Lagrange equations are
$8yz = \lambda \frac{x}{4}$ $8xz = \lambda \frac{y}{9}$ $8xy = \lambda z$
Multiply the first by x, the second by y and the third by z and equate:
$8xyz = \lambda \frac{x^2}{4} = \lambda \frac{y^2}{9} = \lambda z^2$ So $x^2 = 4z^2$ $y^2 = 9z^2$ Plug into the constraint:
$\frac{x^2}{8} + \frac{y^2}{18} + \frac{z^2}{2} = \frac{z^2}{2} + \frac{z^2}{2} + \frac{z^2}{2} = \frac{3z^2}{2} = 6$ $z^2 = 4$
$z = 2$ $x = 2z = 4$ $y = 3z = 6$ $V = 8 \cdot 4 \cdot 6 \cdot 2 = 384$