Name________________________________ Section:__________

MATH 221 Exam 2, Version C Fall 2023

502,503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-8	/48	10	/20
9	/36	Total	/104

1. Consider the function $z = f(x,y) = xy^4$. Its y-trace at $x = 3$ is the intersection of the graph of $z = f(x,y)$ and the plane $x = 3$. Find the slope of this y-trace at $y = 2$.

a. 8

b. 24

c. 27

d. 54

e. 96 Correct

Solution: The slope of the y-trace is the y-partial derivative Here it is $f_y(x,y) = 4xy^3$. So the slope of the y-trace with $x = 3$ at $y = 2$ is $f_y(3,2) = 4 \bullet 3 \bullet 2^3 = 96$.

2. Consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2}$.

Which of the following paths of approach gives a different value of the limit?

a. $y = x^2$ & $x \to 0$

b. $y = x^3 + x^2$ & $x \to 0$

c. $y = x^3 + x^4$ & $x \to 0$ Correct

d. $y = x^4$ & $x \to 0$

e. They are all equal.

Hint: Don't bother multiplying out any quadratic.

Solution:

$$\lim_{\substack{y=x^2\\x\to0}} \frac{x^3y}{x^6+y^2} = \lim_{x\to0} \frac{x^3x^2}{x^6+x^4} = \lim_{x\to0} \frac{x}{x^2+1} = 0$$

$$\lim_{\substack{y=x^3+x^2\\x\to0}} \frac{x^3y}{x^6+y^2} = \lim_{x\to0} \frac{x^3(x^3+x^2)}{x^6+(x^3+x^2)^2} = \lim_{x\to0} \frac{(x^2+x)}{x^2+(x+1)^2} = 0$$

$$\lim_{\substack{y=x^3+x^4\\x\to0}} \frac{x^3y}{x^6+y^2} = \lim_{x\to0} \frac{x^3(x^3+x^4)}{x^6+(x^3+x^4)^2} = \lim_{x\to0} \frac{1+x}{1+(1+x)^2} = \frac{1}{2}$$

$$\lim_{\substack{y=x^4\\x\to0}} \frac{x^3y}{x^6+y^2} = \lim_{x\to0} \frac{x^3x^4}{x^6+x^8} = \lim_{x\to0} \frac{x}{1+x^2} = 0$$

3. Find the plane tangent to the graph of the function $z = \dfrac{x^3}{y^2}$ at the point $(2,1)$.

Its z-intercept is

a. $c = -32$

b. $c = -8$

c. $c = 0$ Correct

d. $c = 8$

e. $c = 32$

Solution: $f(x,y) = \dfrac{x^3}{y^2}$ $\quad f(2,1) = 8$

$f_x(x,y) = \dfrac{3x^2}{y^2} \quad f_x(2,1) = 12 \quad f_y(x,y) = -\dfrac{2x^3}{y^3} \quad f_y(2,1) = -16$

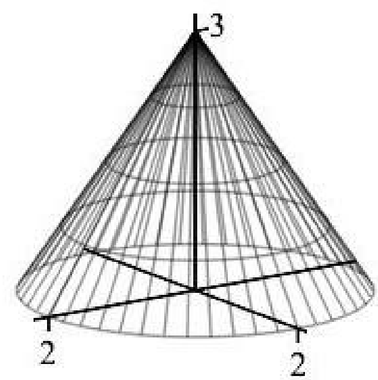
$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 8 + 12(x-2) - 16(y-1)$

$= 12x - 16y + 8 - 24 + 16 = 12x - 16y \qquad c = 0$

4. The volume of a cone is $V = \dfrac{1}{12}\pi D^2 H$ where D is the diameter and H is the height.

Currently, $D = 4$ cm, $H = 3$ cm and $V = 4\pi$ cm.

If V and H increase by $\Delta V = 0.6\pi$ cm and $\Delta H = 0.3$ cm, use the linear approximation to approximate how much D changes.



a. $\Delta D \approx 0.1$ cm Correct

b. $\Delta D \approx 0.2$ cm

c. $\Delta D \approx 0.3$ cm

d. $\Delta D \approx 0.4$ cm

e. $\Delta D \approx 0.6$ cm

Solution: The change in V is approximately its differential:

$$\Delta V \approx dV = \frac{\partial V}{\partial D} dD + \frac{\partial V}{\partial H} dH = \frac{1}{6}\pi D H\, dD + \frac{1}{12}\pi D^2\, dH$$

We plug in numbers and solve for $\Delta D = dD$:

$0.6\pi = \frac{1}{6}\pi 4 \cdot 3 \cdot \Delta D + \frac{1}{12}\pi 16 \cdot 0.3 = 2\pi\Delta D + .4\pi \qquad 0.2 = 2\Delta D \qquad \Delta D = 0.1$ cm

5. Let $f(x,y) = xe^{xy}$. Compute $f_{xy}(2, \ln 2)$.

a. $16 + 16\ln 2$ Correct

b. $4 + 4\ln 2$

c. $4 - 4\ln 2$

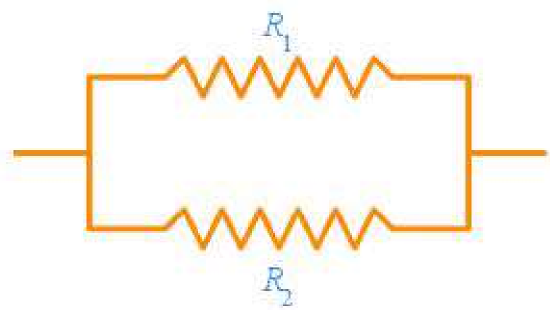
d. $2 + 2\ln 2$

e. $2 - 2\ln 2$

Solution: It's easier to compute f_{yx}. $f_y = x^2 e^{xy}$ $f_{yx} = 2xe^{xy} + x^2 y e^{xy}$
$f_{yx}(2, \ln 2) = 4e^{2\ln 2} + 4\ln 2 e^{2\ln 2} = 4 \cdot 4 + 4\ln 2 \cdot 4 = 16 + 16\ln 2$

6. If two resistors, with resistances R_1 and R_2 are connected in parallel, then the total resistance R satisfies:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R = \frac{R_1 R_2}{R_1 + R_2}$$



Currently $R_1 = 200$ ohms, $R_2 = 400$ ohms and $R = \frac{400}{3}$ ohms. If R_1 and R_2 are increasing at $\frac{dR_1}{dt} = 1.8 \frac{\text{ohms}}{\text{min}}$ and $\frac{dR_2}{dt} = 0.9 \frac{\text{ohms}}{\text{min}}$, at what rate is R changing?

a. $\frac{dR}{dt} = 0.1 \frac{\text{ohms}}{\text{min}}$

b. $\frac{dR}{dt} = 0.3 \frac{\text{ohms}}{\text{min}}$

c. $\frac{dR}{dt} = 0.6 \frac{\text{ohms}}{\text{min}}$

d. $\frac{dR}{dt} = 0.9 \frac{\text{ohms}}{\text{min}}$ Correct

e. $\frac{dR}{dt} = 1.8 \frac{\text{ohms}}{\text{min}}$

Solution: We take the time derivative of the relation:

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{R_1{}^2}\frac{dR_1}{dt} - \frac{1}{R_2{}^2}\frac{dR_2}{dt}$$

and solve for $\frac{dR}{dt}$ and plug in numbers:

$$\frac{dR}{dt} = \frac{R^2}{R_1{}^2}\frac{dR_1}{dt} + \frac{R^2}{R_2{}^2}\frac{dR_2}{dt} = \left(\frac{400}{3}\right)^2\left(\frac{1}{200}\right)^2 1.8 + \left(\frac{400}{3}\right)^2\left(\frac{1}{400}\right)0.9$$

$$= \left(\frac{2}{3}\right)^2 1.8 + \left(\frac{1}{3}\right)^2 0.9 = 0.9 \frac{\text{ohms}}{\text{min}}$$

7. Find the equation of the plane tangent to $x^2 z \sin y = 2\sqrt{2}$ at the point $P = \left(2, \frac{\pi}{4}, 1\right)$.
It's z-intercept is:

a. $c = 3 + \frac{\pi}{4}$ Correct

b. $c = 1 + \frac{\pi}{4}$

c. $c = 3 + \frac{\pi}{2}$

d. $c = 2 + \frac{\pi}{2}$

e. $c = 1 + \frac{\pi}{4}$

Solution: Let $F(x,y,x) = x^2 z \sin y$. Then $\vec{\nabla}F = \langle 2xz\sin y, x^2 z\cos y, x^2 \sin y\rangle$ and
$\vec{N} = \vec{\nabla}F\Big|_P = \left\langle \frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}}, \frac{4}{\sqrt{2}} \right\rangle$. So the plane is $\vec{N} \bullet X = \vec{N} \bullet P$ or
$\frac{4}{\sqrt{2}}x + \frac{4}{\sqrt{2}}y + \frac{4}{\sqrt{2}}z = \frac{4}{\sqrt{2}}2 + \frac{4}{\sqrt{2}}\frac{\pi}{4} + \frac{4}{\sqrt{2}}$ or $x + y + z = 3 + \frac{\pi}{4}$
The z-intercept occurs when $x = y = 0$. So the z-intercept is $c = 3 + \frac{\pi}{4}$.

8. The point $(x,y) = (2,1)$ is a critical point of the function $f(x,y) = 12xy - x^3 - 8y^3$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum

b. Local Maximum Correct

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $f_x = 12y - 3x^2$ $\quad f_y = 12x - 24y^2$
We check: $f_x(2,1) = 24 - 24 = 0$ $\quad f_y(2,1) = 24 - 24 = 0$, Yes, it's a critical point.
$f_{xx} = -6x$ $\quad f_{yy} = -48y$ $\quad f_{xy} = 12$
$f_{xx}(2,1) = -12 < 0$ $\quad f_{yy}(2,1) = -48 < 0$ $\quad f_{xy}(2,1) = 12$
$D = f_{xx}f_{yy} - f_{xy}{}^2 = (-12)(-48) - 12^2 > 0$ Local Maximum

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (36 pts) Obi-Two is flying the Centurion Eagle through the Force, Desperation and Luck fields. The Force, F, depends on the Desperation, D, and Luck, L, by the relation: $F = 3DL^2$. Currently, the Desperation and Luck and their gradients are:

$$D = 3 \qquad \vec{\nabla} D = \langle 3,0,1 \rangle$$

$$L = 2 \qquad \vec{\nabla} L = \langle 1,2,0 \rangle$$

a. (23 pts) Obi-Two's current velocity is $\vec{v} = \langle 1,2,3 \rangle$. Find the rate that Obi-Two sees the Force changing.

Solution: The partial derivatives of $\vec{F}$ are:

$$\frac{\partial F}{\partial D} = 3L^2 = 3 \cdot 2^2 = 12 \qquad \frac{\partial F}{\partial L} = 6DL = 6 \cdot 3 \cdot 2 = 36$$

The derivatives of x, y, z, D and L are components of the velocity and the gradients. So by the Chain Rule, the derivative of F is:

$$\frac{dF}{dt} = \frac{\partial F}{\partial D}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) + \frac{\partial F}{\partial L}\left(\frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} + \frac{\partial L}{\partial z}\frac{dz}{dt}\right)$$

$$= 12(3 \cdot 1 + 1 \cdot 3) + 36(1 \cdot 1 + 2 \cdot 2) = 72 + 180 = 252$$

b. (13 pts) In what direction should Obi-Two travel to increase the Force as fast as possible? HINT: Compute each (x,y,z) partial derivative separately.

Solution: The partial derivatives of $\vec{F}$ are:

$$\frac{\partial F}{\partial D} = 3L^2 = 3 \cdot 2^2 = 12 \qquad \frac{\partial F}{\partial L} = 6DL = 6 \cdot 3 \cdot 2 = 36$$

The components of the gradient are the (x,y,z) partial derivatives of $\vec{F}$:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial x} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial x} = 12 \cdot 3 + 36 \cdot 1 = 72$$

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial y} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial y} = 12 \cdot 0 + 36 \cdot 2 = 72$$

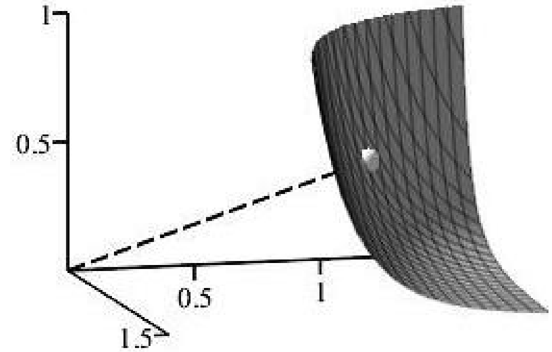
$$\frac{\partial F}{\partial z} = \frac{\partial F}{\partial D}\frac{\partial D}{\partial z} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial z} = 12 \cdot 1 + 36 \cdot 0 = 12$$

The Force increases fastest in the direction $\vec{\nabla} F = \langle 72,72,12 \rangle$.
Alternatively:

$$\vec{\nabla} F = \frac{\partial F}{\partial D}\vec{\nabla} D + \frac{\partial F}{\partial L}\vec{\nabla} L$$

10. (20 pts) Find the point on the surface $z = \frac{1}{4x^2y^4}$ in the 1^{st} octant which is closest to the origin.

HINT: Write the constraint as $g = x^2y^4z = \frac{1}{4}$.



Solution: We minimize the square of the distance, $f = D^2 = x^2 + y^2 + z^2$.
The constraint is $g = x^2y^4z = \frac{1}{4}$. The gradients are:

$\vec{\nabla}V = \langle 2x, 2y, 2z \rangle$ $\quad\quad$ $\vec{\nabla}g = \langle 2xy^4z, 4x^2y^3z, x^2y^4 \rangle$ $\quad\quad$ The Lagrange equations are:

$2x = \lambda 2xy^4z$ $\quad\quad$ $2y = \lambda 4x^2y^3z$ $\quad\quad$ $2z = \lambda x^2y^4$

Multiply the first by $\frac{x}{2}$, the second by $\frac{y}{4}$ and the third by z and equate:

$\lambda x^2y^4z = x^2 = \frac{y^2}{2} = 2z^2$ $\quad\quad$ So $x^2 = 2z^2$ $\quad$ $y^2 = 4z^2$ $\quad\quad$ Plug into the constraint:

$x^2y^4z = 2z^2 16z^4z = 32z^7 = \frac{1}{4}$ $\quad\Rightarrow\quad$ $z^7 = \frac{1}{128}$ $\quad\Rightarrow$

$z = \frac{1}{2}$ $\quad\quad$ $x = \sqrt{2}\,z = \frac{\sqrt{2}}{2}$ $\quad\quad$ $y^2 = 4z^2 = 4\left(\frac{1}{2}\right)^2 = 1$ $\quad\quad$ $(x,y,z) = \left(\frac{\sqrt{2}}{2}, 1, \frac{1}{2}\right)$