

Name _____ Section: _____

MATH 221 Exam 3, Version C Fall 2023

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Multiple Choice: (6 points each. No part credit.)

1-8	/48	10	/16
9	/16	11	/24
		Total	/104

1. Find the divergence of the vector field $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$ and evaluate the divergence at $P = (1, 2, 3)$.

- a. -2
- b. 20
- c. 22 Correct
- d. 23
- e. 24

Solution: $\vec{\nabla} \cdot F = 2xy + 2yz + 2zx \quad \vec{\nabla} \cdot F|_P = 4 + 12 + 6 = 22$

2. Find the curl of the vector field $\vec{F} = \langle x^2y, y^2z, z^2x \rangle$ and evaluate the curl at $P = (1, 2, 3)$.

- a. $\langle -4, -9, -1 \rangle$ Correct
- b. $\langle -4, 9, -1 \rangle$
- c. $\langle 2, -12, 9 \rangle$
- d. $\langle 2, 12, 9 \rangle$
- e. $\langle 1, -4, 9 \rangle$

Solution: $\vec{\nabla} \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ x^2y & y^2z & z^2x \end{vmatrix} = \hat{i}(-y^2) - \hat{j}(z^2) + \hat{k}(-x^2) \quad \vec{\nabla} \times F|_P = \langle -4, -9, -1 \rangle$

3. Let f be a scalar potential for $\vec{F} = \langle yz + 2x, xz + 2y, xy + 2z \rangle$. Compute $f(1, 2, 3) - f(0, 0, 0)$. (Note: The subtraction cancels off the arbitrary constant.)

- a. -2
- b. 20 Correct
- c. 22
- d. 23
- e. 24

Solution: By inspection, $f = xyz + x^2 + y^2 + z^2$.
Then $f(1, 2, 3) - f(0, 0, 0) = (6 + 1 + 4 + 9) - (0) = 20$

4. Use a Riemann sum with 6 squares evaluated at the center of each square to estimate the volume of the solid over the rectangle $[1, 7] \times [2, 6]$ below the surface $f = x^2 + y^2$.
- 130
 - 214
 - 520
 - 856 Correct
 - 872

Solution: The volume is $V = \iint f dA$. The area of each square is $\Delta A = \Delta x \Delta y = 2 \cdot 2 = 4$.

The centers are:

$$(2, 3), (2, 5), (4, 3), (4, 5), (6, 3), (6, 5)$$

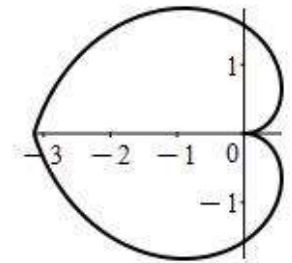
The function values are:

$$f(2, 3) = 13, f(2, 5) = 29, f(4, 3) = 25, f(4, 5) = 41, f(6, 3) = 45, f(6, 5) = 61$$

The Riemann sum is:

$$V = \iint f dA \approx \sum_{k=1}^6 f(x_i^*, y_j^*) \Delta A = (13 + 29 + 25 + 41 + 45 + 61) 4 = 856$$

5. Find the area inside the heart which in polar coordinates is the spiral $r = |\theta|$ for $-\pi \leq \theta \leq \pi$.
HINT Double the area inside half the spiral.



- $\frac{\pi^3}{6}$
- $\frac{\pi^3}{3}$ Correct
- $\frac{\pi^3}{2}$
- $\frac{\pi^4}{4}$
- $\frac{\pi^4}{2}$

Solution: $A = \iint 1 dA = 2 \int_0^\pi \int_0^\theta r dr d\theta = 2 \int_0^\pi \left[\frac{r^2}{2} \right]_0^\theta d\theta = \int_0^\pi \theta^2 d\theta = \left[\frac{\theta^3}{3} \right]_0^\pi = \frac{\pi^3}{3}$

6. Find mass of the solid below $z = 25 - x^2 - y^2$
above the xy -plane inside the cylinder $x^2 + y^2 = 9$
if the volume density is $\delta = z$.



- a. $\frac{\pi}{6}(25^3 - 16^3)$ Correct
b. $\frac{\pi}{6}(25^3 + 16^3)$
c. $\pi\left(25^2 \cdot 3 - 50 \cdot 3^2 + \frac{3^5}{5}\right)$
d. $\pi\left(25^2 \cdot 3 + 50 \cdot 3^2 + \frac{3^5}{5}\right)$
e. $\pi\left(25 \frac{3^2}{2} - \frac{3^4}{4}\right)$

Solution: In cylindrical coordinates, the top is $z = 25 - r^2$, the sides are $r = 3$ and the volume differential is $dV = r dr d\theta dz$.

$$\begin{aligned} M &= \iiint_V \delta dV = \int_0^{2\pi} \int_0^3 \int_0^{25-r^2} z r dz dr d\theta = 2\pi \int_0^3 \left[\frac{z^2}{2} \right]_0^{25-r^2} r dr = \pi \int_0^3 (25 - r^2)^2 r dr \\ &= -\frac{\pi}{2} \int_{25}^{16} u^2 du = -\frac{\pi}{2} \left[\frac{u^3}{3} \right]_{25}^{16} = -\frac{\pi}{6} (16^3 - 25^3) = \frac{\pi}{6} (25^3 - 16^3) \end{aligned}$$

7. Find the Jacobian factor for the 3D coordinate system:

$$(x, y, z) = \vec{R}(u, v, w) = (vw, uw, uv) \quad \text{with } u > 0, \quad v > 0, \quad w > 0$$

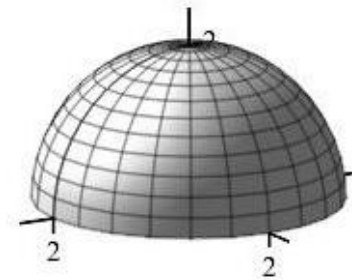
- a. uvw
b. $2uvw$ Correct
c. $u + v + w$
d. $2u + 2v + 2w$
e. $vw + uw + uv$

Solution: The Jacobian determinant is

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{vmatrix} = 0|*| - w \begin{vmatrix} w & u \\ v & 0 \end{vmatrix} + v \begin{vmatrix} w & 0 \\ v & u \end{vmatrix} = -w(-uv) + v(wu) = 2uvw$$

Since this is positive $J = 2uvw$.

8. Find the average value of the function $f = x^2 + y^2 + z^2$ on the solid hemisphere $0 \leq z \leq \sqrt{4 - x^2 - y^2}$.



- a. $\frac{5}{5}$
 b. $\frac{4}{5}\pi$
 c. $\frac{12}{5}$ Correct
 d. $\frac{16}{3}\pi$
 e. $\frac{64}{5}\pi$

Solution: The radius of the sphere is $\rho = 2$. The volume of the hemisphere is $V = \frac{2}{3}\pi\rho^3 = \frac{16}{3}\pi$.

In spherical coordinates the function is $f = \rho^2$. So the integral of the function is:

$$\iiint f dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \rho^2 \sin \phi d\rho d\phi d\theta = \left[\theta \right]_0^{2\pi} \left[-\cos \phi \right]_0^{\pi/2} \left[\frac{\rho^5}{5} \right]_0^2 = (2\pi)(1) \frac{32}{5} = \frac{64\pi}{5}$$

So the average of f is $f_{\text{ave}} = \frac{1}{V} \iiint f dV = \frac{3}{16\pi} \frac{64\pi}{5} = \frac{12}{5}$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (16 points) Consider a plate bounded by $y = 4 - x^2$ and the x -axis with surface density $\delta = x^2$.
- a. (8 pts) Find the mass of a plate.

Solution: The mass is:

$$\begin{aligned} M &= \iint \delta dA = \int_{-2}^2 \int_0^{4-x^2} x^2 dy dx = \int_{-2}^2 \left[x^2 y \right]_{y=0}^{4-x^2} dx = \int_{-2}^2 x^2(4-x^2) dx \\ &= \left[\frac{4x^3}{3} - \frac{x^5}{5} \right]_{-2}^2 = 2 \left(\frac{32}{3} - \frac{32}{5} \right) = 64 \frac{2}{15} = \frac{128}{15} \end{aligned}$$

- b. (8 pts) Find the center of mass of a plate.

Solution: By symmetry $\bar{x} = 0$. The y -moment is:

$$\begin{aligned} M_y &= \iint y \delta dA = \int_{-2}^2 \int_0^{4-x^2} y x^2 dy dx = 3 \int_{-2}^2 \left[x^2 \frac{y^2}{2} \right]_{y=0}^{4-x^2} dx = \frac{1}{2} \int_{-2}^2 x^2(4-x^2)^2 dx \\ &= \frac{1}{2} \int_{-2}^2 x^2(16 - 8x^2 + x^4) dx = \frac{1}{2} \left[16 \frac{x^3}{3} - 8 \frac{x^5}{5} + \frac{x^7}{7} \right]_{-2}^2 = \left(\frac{27}{3} - \frac{28}{5} + \frac{27}{7} \right) \\ &= 27 \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right) = \frac{1024}{105} \end{aligned}$$

$$\text{So } \bar{y} = \frac{M_y}{M} = \frac{1024}{105} \frac{15}{128} = \frac{8}{7}$$

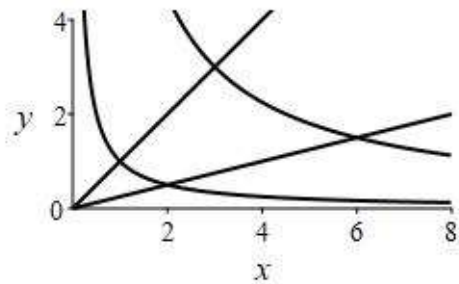
10. (16 points) Compute $\iint_D x^2 dA$ over the diamond

shaped region in the 1st quadrant bounded by

$$y = \frac{1}{x} \quad y = \frac{9}{x} \quad y = x \quad y = \frac{1}{4}x$$

HINT: Use the curvilinear coordinates

$$x = uv \quad y = \frac{v}{u}.$$



a. (5 pts) Find the Jacobian factor.

$$\text{Solution: } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & -\frac{v}{u^2} \\ u & \frac{1}{u} \end{vmatrix} = \left| \frac{v}{u} - -\frac{v}{u} \right| = \frac{2v}{u}$$

b. (1 pts) Express the integrand in terms of the coordinates.

$$\text{Solution: } x^2 = u^2 v^2.$$

c. (4 pts) Substitute $x = uv$ and $y = \frac{v}{u}$ into the boundaries to express them in terms of u and v .

$$\text{Solution: } xy = uv \frac{v}{u} = v^2 \quad \text{So two boundaries are } v^2 = 1 \quad \text{and } v^2 = 9 \quad \text{or } v = 1 \quad \text{and } v = 3.$$

$$\frac{y}{x} = \frac{v}{u} \frac{1}{uv} = \frac{1}{u^2}. \quad \text{So two boundaries are } \frac{1}{u^2} = 1 \quad \text{and } \frac{1}{u^2} = \frac{1}{4} \quad \text{or } u = 1 \quad \text{and } u = 2.$$

d. (6 pts) Compute the integral.

Solution:

$$\iint_D x^2 dA = \int_1^3 \int_1^2 u^2 v^2 \cdot \frac{2v}{u} du dv = \int_1^3 2v^3 dv \int_1^2 u du = \left[\frac{v^4}{2} \right]_1^3 \left[\frac{u^2}{2} \right]_1^2 = \frac{81-1}{2} \frac{4-1}{2} = 60$$

11. (24 points) Consider the cone surface $z = 2\sqrt{x^2 + y^2}$ for $z \leq 6$ which may be parametrized by

$$\vec{R}(r, \theta) = \langle r \cos \theta, r \sin \theta, 2r \rangle$$

- a. (6 pts) Find the tangent vectors:

Solution:

$$\vec{e}_r = \langle \hat{i} \cos \theta, \hat{j} \sin \theta, \hat{k} \cdot 2 \rangle = \langle \cos \theta, \sin \theta, 2 \rangle$$

$$\vec{e}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

- b. (3 pts) Find the normal vector oriented down and out:

Solution: $\vec{N} = \hat{i}(-2r \cos \theta) - \hat{j}(2r \sin \theta) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta) = \langle -2r \cos \theta, -2r \sin \theta, r \rangle$
 This is up and in. So we reverse it: $\vec{N} = \langle 2r \cos \theta, 2r \sin \theta, -r \rangle$

- c. (2 pts) Find the length of the normal vector:

Solution: $|\vec{N}| = \sqrt{4r^2 \cos^2 \theta + 4r^2 \sin^2 \theta + r^2} = \sqrt{5} r$

- d. (4 pts) Find the mass of the cone if the mass density is $\delta = \sqrt{x^2 + y^2}$.

Solution: $M = \iiint \delta \, dS = \iiint \delta |\vec{N}| \, dr \, d\theta = \int_0^{2\pi} \int_0^3 r \sqrt{5} r \, dr \, d\theta = 2\pi \sqrt{5} \left[\frac{r^3}{3} \right]_0^3 = 18\sqrt{5} \pi$

- e. (3 pts) Find the **curl** of the vector field $\vec{F} = \langle yz, -xz, z^2 \rangle$ in rectangular coordinates:

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & z^2 \end{vmatrix} = \hat{i}(0 + x) - \hat{j}(0 - y) + \hat{k}(-z - z) = \langle x, y, -2z \rangle$

- f. (2 pts) Evaluate the **curl** of \vec{F} on the cone:

Solution: $\vec{\nabla} \times \vec{F} \Big|_{\vec{R}} = \langle x, y, -2z \rangle = \langle r \cos \theta, r \sin \theta, -4r \rangle$

- g. (4 pts) Find the flux of the **curl** of \vec{F} down and out of the cone:

Solution: $\vec{\nabla} \times \vec{F} \Big|_{\vec{R}} \cdot \vec{N} = 2r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + 4r^2 = 6r^2 \quad z = 2r \leq 6 \quad r \leq 3$

$$\iint_C (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_0^{2\pi} \int_0^3 6r^2 \, dr \, d\theta = 2\pi \left[2r^3 \right]_0^3 = 108\pi$$