Name________________________ Section:________

MATH 221 Final Exam, Version B Fall 2023

502,503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1-12	/60	14	/10
13	/10	15	/24
		Total	/104

1. Which of the following lines lies in the plane: $2x - y - z = 0$?

a. $(x,y,z) = (1,2,3) + t(1,1,1)$

b. $(x,y,z) = (3,2,1) + t(1,1,1)$

c. $(x,y,z) = (2,1,3) + t(1,1,1)$ Correct

d. $(x,y,z) = (3,1,2) + t(1,1,1)$

e. $(x,y,z) = (1,3,2) + t(1,1,1)$

Solution: The starting point must be on the plane. So we set $t = 0$ and try each point. Only $(x,y,z) = (2,1,3)$ satisfies $2x - y - z = 0$

2. Find the arc length of the curve $\vec{r}(t) = (2t^2, t^3)$ between $(0,0)$ and $(2,1)$.

a. $\frac{31}{27}$

b. $\frac{61}{27}$ Correct

c. $\frac{91}{27}$

d. $\frac{31}{9}$

e. $\frac{61}{9}$

Solution: $\vec{v} = (4t, 3t^2)$ $|\vec{v}| = \sqrt{16t^2 + 9t^4} = t\sqrt{16 + 9t^2}$ You may need $u = 16 + 9t^2$ $du = 18t\,dt$.

$$L = \int ds = \int |\vec{v}|\,dt = \int_0^1 t\sqrt{16 + 9t^2}\,dt = \frac{1}{27}\left[(16 + 9t^2)^{3/2}\right]_0^1 = \frac{1}{27}(25^{3/2} - 16^{3/2}) = \frac{1}{27}(125 - 64) = \frac{61}{27}$$

3. Find the point where the line $\frac{x-4}{3} = \frac{y-4}{2} = z-4$ intersects the plane $3x+2y+z = 10$.
At this point $x+y+z =$

a. 0
b. 2
c. 4
d. 6 Correct
e. They do not intersect.

Solution: We convert the line to parametric form, plug into the plane and solve for t:
$x = 4+3t \quad y = 4+2t \quad z = 4+t \qquad 3(4+3t)+2(4+2t)+(4+t) = 10$
$24+14t = 10 \quad 14t = -14 \quad t = -1 \quad (x,y,z) = (4+3t,4+2t,4+t) = (1,2,3)$
So $x+y+z = 6$

4. Find the equation of the plane tangent to the graph of $z = 3x^2y-2y^3$ at the point $(2,1)$.
Its z-intercept is

a. -20 Correct
b. -14
c. 14
d. 20
e. 40

Solution: $f(x,y) = 3x^2y-2y^3 \qquad f_x(x,y) = 6xy \qquad f_y(x,y) = 3x^2-6y^2$
$f(2,1) = 3\cdot 2^2-2 = 10 \qquad f_x(2,1) = 6\cdot 2 = 12 \qquad f_y(2,1) = 3\cdot 2^2-6 = 6$
$z = f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1) = 14+12(x-2)+18(y-1)$
z-intercept $= 10+12(-2)+6(-1) = -20$

5. Find the equation of the line perpendicular to the graph of $x^3y^2z-2x^2z^2 = 10$ at the point $(1,3,2)$.
This line intersects the xy-plane at:

a. $\left(-\frac{19}{3},2,0\right)$
b. $\left(2,\frac{19}{3},0\right)$
c. $(-75,-21,0)$ Correct
d. $\left(\frac{19}{3},-2,0\right)$
e. $(21,-75,0)$

Solution: $f(x,y,z) = x^3y^2z-2x^2z^2 \qquad \vec{\nabla}f = (3x^2y^2z-4xz^2, 2x^3yz, x^3y^2-4x^2z)$
$\vec{N} = \vec{\nabla}f(1,3,2) = \left(3(3)^2(2)-4(2)^2, 2(3)(2), (3)^2-4(2)\right) = (38,12,1)$
$X = P+t\vec{N} \qquad (x,y,z) = (1,3,2)+t(38,12,1) = (1+38t,3+12t,2+t)$
This line intersects the xy-plane when $z = 2+t = 0$ or $t = -2$ or at $(x,y,z) = (-75,-21,0)$

6. The volume of a cone is $V = \frac{1}{3}\pi r^2 h.$

The radius r is currently 3 cm and is increasing at 2 cm/sec.

The height h is currently 4 cm and is decreasing at 1 cm/sec.

Is the volume increasing or decreasing and at what rate?

a. decreasing at 19π cm$^3/\text{sec}$

b. decreasing at 13π cm$^3/\text{sec}$

c. neither increasing nor decreasing

d. increasing at 19π cm$^3/\text{sec}$

e. increasing at 13π cm$^3/\text{sec}$ Correct

Solution: $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt} = \frac{2}{3}\pi 3 \cdot 4(2) + \frac{1}{3}\pi 9(-1)$
$= (16-3)\pi = 13\pi$ Since this is positive, the volume is increasing.

7. Compute the line integral $\int -y\,dx + x\,dy$ along the parabola $y = x^2$ from $(1,1)$ to $(2,4)$.

HINT: Parametrize the curve.

a. $\frac{7}{3}$ Correct

b. $\frac{5}{3}$

c. $\frac{1}{3}$

d. 1

e. 3

Solution: $\vec{r}(t) = (t,t^2)$ for $1 \le t \le 2$. $\vec{v} = (1,2t)$ This points right and up as needed.
$\vec{F} = (-y,x) = (-t^2,t)$ $\vec{F} \cdot \vec{v} = -t^2 + 2t^2 = t^2$
$\int y\,dx - x\,dy = \int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{v}\,dt = \int_1^2 t^2\,d\theta = \left[\frac{t^3}{3}\right]_1^2 = \frac{7}{3}$

8. Compute $\int_{\vec{r}} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x+y+z, 2y+x+z, 2z+x+y)$ along the curve
$\vec{r}(t) = (t\cos t, t\sin t, t\,e^{t/\pi})$ between $t = 0$ and $t = \pi$.

HINT: Find a scalar potential.

a. $\pi^2(1+e^2-e)$ Correct

b. $\pi^2(1+e^2-2e)$

c. $\pi^2(1+e^2+e)$

d. $\pi^2(1+e^2+2e)$

e. $\pi^2(1+e^2)$

Solution: $\vec{F} = \vec{\nabla} f$ for $f = x^2+y^2+z^2+xy+xz+yz$
$A = \vec{r}(0) = (0,0,0)$ $B = \vec{r}(\pi) = (-\pi,0,\pi e)$
By the FTCC. $\int_A^B \vec{F} \cdot d\vec{s} = \int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = \pi^2 + \pi^2 e^2 - \pi^2 e = \pi^2(1+e^2-e)$

9. Compute $\oint_C \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-x^2y + x^3 - y^3, xy^2 + x^3 - y^3)$
counterclockwise around the circle $x^2 + y^2 = 9$.
HINT: Use a theorem.

a. 324π

b. 162π Correct

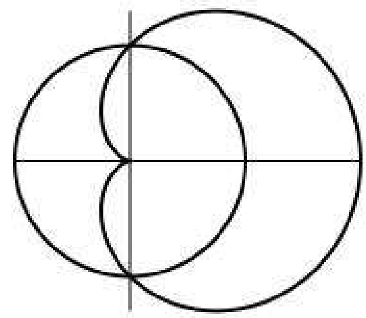
c. 144π

d. 72π

e. 36π

Solution: We identify $P = -x^2y + x^3 - y^3$ and $Q = xy^2 + x^3 - y^3$.
So $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = (y^2 + 3x^2) - (-x^2 - 3y^2) = 4(x^2 + y^2) = 4r^2$.
By Green's Theorem: $\oint_C \vec{F} \cdot d\vec{s} = \iint \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}\, dA = \int_0^{2\pi} \int_0^3 4r^2\, r\, dr\, d\theta = 2\pi[r^4]_0^3 = 162\pi$

10. Find the area inside the cardioid $r = 1 + \cos\theta$
but outside the circle $r = 1$.



a. $\frac{\pi}{4}$

b. $\frac{\pi}{2}$

c. $2 - \frac{\pi}{4}$

d. $2 + \frac{\pi}{4}$ Correct

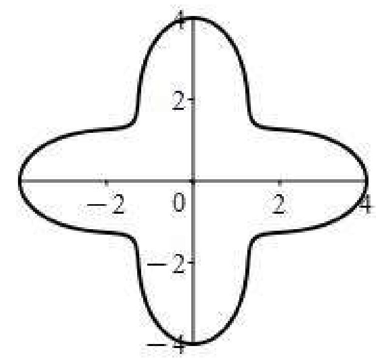
e. $2 - \frac{\pi}{2}$

Solution: $A = \iint 1\, dA = \int_{-\pi/2}^{\pi/2} \int_1^{1+\cos\theta} r\, dr\, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^2}{2}\right]_1^{1+\cos\theta} d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left[(1 + \cos\theta)^2 - 1\right] d\theta$

$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} [2\cos\theta + \cos^2\theta]\, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left[2\cos\theta + \frac{1 + \cos 2\theta}{2}\right] d\theta = \frac{1}{2}\left[2\sin\theta + \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)\right]_{-\pi/2}^{\pi/2}$

$= 2\sin\frac{\pi}{2} + \frac{1}{2}\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) = 2 + \frac{\pi}{4}$

11. Compute $\oint \vec{\nabla} f \cdot d\vec{s}$ counterclockwise once around the polar curve $r = 3 + \cos(4\theta)$ for the function $f(x,y) = x^2 y$.



a. 2π

b. 4π

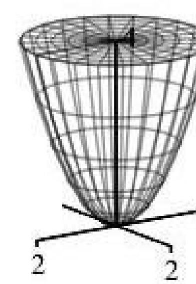
c. 6π

d. 8π

e. 0 Correct

Solution: For a closed curve, the starting and finishing points are the same, $A = B$. So by the FTCC, $\oint \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = 0$ for any function f.

12. Consider the parabolic surface P given by $z = x^2 + y^2$ for $z \le 4$ with normal pointing up and in, the disk D given by $x^2 + y^2 \le 4$ and $z = 4$ with normal pointing up, and the volume V between them. For a certain vector field $\vec{F}$ we have:



$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = 14 \quad \text{and} \quad \iint_D \vec{F} \cdot d\vec{S} = 3$$

Compute $\iint_P \vec{F} \cdot d\vec{S}$.

a. 17

b. 11

c. 8

d. -11 Correct

e. -17

Solution: By Gauss' Theorem: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_D \vec{F} \cdot d\vec{S} - \iint_P \vec{F} \cdot d\vec{S}$

The minus sign reverses the orientation of P to point outward. Thus

$$\iint_P \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} - \iiint_V \vec{\nabla} \cdot F\, dV = 3 - 14 = -11$$

Work Out: (Points shown. Part credit possible. Show all work.)

13. (10 points) Find 3 numbers a, b and c whose sum is 12 for which $ab + 2ac + 3bc$ is a maximum.

You do not need to show it is a maximum.

Solution: We maximize $f = ab + 2ac + 3bc$ subject to the constraint $a + b + c = 12$.

Solve the constraint: $c = 12 - a - b$ Substitute into the function:

$$f = ab + 2a(12 - a - b) + 3b(12 - a - b) = 24a + 36b - 2a^2 - 4ab - 3b^2$$

Set the partial derivatives equal to zero and solve:

$$f_a = 24 - 4a - 4b = 0 \qquad 4a + 4b = 24$$
$$f_b = 36 - 4a - 6b = 0 \qquad 4a + 6b = 36$$

$$2b = 12 \qquad b = 6 \qquad a + b = 6 \qquad a = 6 - b = 0$$

$$c = 12 - a - b = 12 - 0 - 6 = 6 \qquad \boxed{a = 0, \quad b = 6, \quad c = 6}$$

14. (10 points) Find the mass and center of mass of the cylindrical **surface** $x^2 + y^2 = 9$ for $0 \le z \le 2$ with density $\delta = z$. The cylinder may be parametrize as $\vec{R}(\theta, z) = (3\cos\theta, 3\sin\theta, z)$.

Solution:

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta & = & (-3\sin\theta, & 3\cos\theta, & 0) \\ \vec{e}_z & = & (\ \ 0, & 0, & 1) \end{array}$$

$$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(3\cos\theta) - \hat{\jmath}(-3\sin\theta) + \hat{k}(0) = (3\cos\theta, 3\sin\theta, 0)$$

$$\left|\vec{N}\right| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3 \qquad \rho = z$$

$$M = \iint \delta\, dS = \int\int z \left|\vec{N}\right| d\theta\, dz = \int_0^2 \int_0^{2\pi} 3z\, d\theta\, dz = 2\pi 3\left[\frac{z^2}{2}\right]_0^2 = 12\pi$$

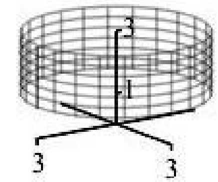
$\bar{x} = \bar{y} = 0$ by symmetry.

$$z\text{-mom} = M_z = \iint z\delta\, dS = \int\int z^2 \left|\vec{N}\right| d\theta\, dz = \int_0^2 \int_0^{2\pi} 3z^2\, d\theta\, dz = 2\pi 3\left[\frac{z^3}{3}\right]_0^2 = 16\pi$$

$$\bar{z} = \frac{M_z}{M} = \frac{M_z}{M} = \frac{16\pi}{12\pi} = \frac{4}{3}$$

15. (24 points) Verify Stokes' Theorem

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$$



for the vector field $\vec{F} = (yz^2, -xz^2, z^3)$ and the cylinder $x^2 + y^2 = 9$ for $1 \le z \le 2$ oriented outward.

Be sure to check orientations. Use the following steps:

Left Hand Side: The cylindrical surface may be parametrized by $\vec{R}(\theta, z) = (3\cos\theta, 3\sin\theta, z)$.

a. Compute the normal vector and check its orientation:

Solution:

$$\begin{array}{rcccc} & & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta & = & (-3\sin\theta, & 3\cos\theta, & 0) \\ \vec{e}_z & = & (\ 0, & 0, & 1) \end{array}$$

$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(3\cos\theta) - \hat{\jmath}(-3\sin\theta) + \hat{k}(0) = (3\cos\theta, 3\sin\theta, 0)$

This correctly points outward.

b. Compute the curl of $\vec{F}$ and evaluate it on the cylinder.

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz^2 & -xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - -2xz) - \hat{\jmath}(0 - 2yz) + \hat{k}(-z^2 - z^2) = (2xz, 2yz, -2z^2)$

$\vec{\nabla} \times \vec{F}\left(\vec{R}(\theta, z)\right) = (6z\cos\theta, 6z\sin\theta, -2z^2)$

c. Compute the dot product of the curl of $\vec{F}$ and the normal:

Solution: $\vec{\nabla} \times \vec{F} \cdot \vec{N} = 18z\cos^2\theta + 18z\sin^2\theta = 18z$

d. Compute the surface integral:

Solution: $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\theta\, dz = \int_1^2 \int_0^{2\pi} 18z\, d\theta\, dz = 2\pi[9z^2]_{z=1}^{2} = 54\pi$

(continued)

Right Hand Side: Let U be the upper circle and L be the lower circle.

e. Parametrize U. Find the velocity and check its orientation:

Solution: $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 2)$ $\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$
This is counterclockwise. We need clockwise. So we reverse it: $\vec{v}(\theta) = (3\sin\theta, -3\cos\theta, 0)$

f. Evaluate $\vec{F} = (yz^2, -xz^2, z^3)$ on the circle and compute its dot product with the velocity:

Solution: $\vec{F}(\vec{r}(\theta)) = (12\sin\theta, -12\cos\theta, 8)$ $\vec{F} \cdot \vec{v} = 36\sin^2\theta + 36\cos^2\theta = 36$

g. Compute the line integral $\oint_U \vec{F} \cdot d\vec{s}$

Solution: $\oint_U \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 36\, d\theta = 72\pi$

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h. Parametrize L. Find the velocity and check its orientation:

Solution: $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 1)$ $\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$
This is counterclockwise as needed.

i. Evaluate $\vec{F} = (yz^2, -xz^2, z^3)$ on the circle and compute its dot product with the velocity:

Solution: $\vec{F}(\vec{r}(\theta)) = (3\sin\theta, -3\cos\theta, 1)$ $\vec{F} \cdot \vec{v} = -9\sin^2\theta - 9\cos^2\theta = -9$

j. Compute the line integral $\oint_L \vec{F} \cdot d\vec{s}$

Solution: $\oint_L \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -9\, d\theta = -18\pi$

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k. Combine $\oint_U \vec{F} \cdot d\vec{s}$ and $\oint_L \vec{F} \cdot d\vec{s}$ to get $\oint_{\partial C} \vec{F} \cdot d\vec{s}$.

Solution: $\oint_{\partial C} \vec{F} \cdot d\vec{s} = \oint_U \vec{F} \cdot d\vec{s} + \oint_L \vec{F} \cdot d\vec{s} = 72\pi - 18\pi = 54\pi$

which agrees with part (d).