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MATH 221 Final Exam Version H Fall 2019
Section 204 Solutions P. Yasskin
Multiple Choice: (5 points each. No part credit.)

1-11	/55	13	/10
12	/15	14	/25
		Total	/105

1. A triangle has vertices $A = (3, 2, 1)$, $B = (3, 3, 2)$ and $C = (4, 4, 2)$. Find the angle at A .
- a. 0°
 - b. 30° Correct Choice
 - c. 45°
 - d. 60°
 - e. 90°

Solution: $\vec{AB} = B - A = (0, 1, 1)$ $\vec{AC} = C - A = (1, 2, 1)$ $|\vec{AB}| = \sqrt{2}$ $|\vec{AC}| = \sqrt{6}$
 $\vec{AB} \cdot \vec{AC} = 0 + 2 + 1 = 3$ $\cos \theta = \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| |\vec{AC}|} = \frac{3}{\sqrt{2} \sqrt{6}} = \frac{\sqrt{3}}{2}$ $\theta = 30^\circ$

2. Find the area of the triangle with vertices $A = (3, 2, 1)$, $B = (3, 3, 2)$ and $C = (4, 4, 2)$.
- a. $2\sqrt{6}$
 - b. $\sqrt{6}$
 - c. $\frac{\sqrt{6}}{2}$
 - d. $\sqrt{3}$
 - e. $\frac{\sqrt{3}}{2}$ Correct Choice

Solution: $\vec{AB} = B - A = \langle 0, 1, 1 \rangle$ $\vec{AC} = C - A = \langle 1, 2, 1 \rangle$
 $\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \hat{i}(1 - 2) - \hat{j}(0 - 1) + \hat{k}(0 - 1) = \langle -1, 1, -1 \rangle$ $A = \frac{\sqrt{3}}{2}$

3. Find the arc length of the curve $\vec{r}(t) = (t^2, 2t, \ln t)$ between $(1, 2, 0)$ and $(e^2, 2e, 1)$.

- a. $e^2 + 1$
- b. $e^2 - 1$
- c. e^2 Correct Choice
- d. $\sqrt{e^2 + 1}$
- e. $\sqrt{e^2 + 1} - 1$

Solution: $|\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(2t)^2 + (2)^2 + (t^{-1})^2} = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = 2t + \frac{1}{t}$

$L = \int_1^e ds = \int_1^e |\vec{v}| dt = \int_1^e \left(2t + \frac{1}{t}\right) dt = \left[t^2 + \ln t\right]_1^e = (e^2 + 1) - (1 - 0) = e^2$

4. Find the average value of the function $f(x, y, z) = x$ along the curve $\vec{r}(t) = (t^2, 2t, \ln t)$ between $(1, 2, 0)$ and $(e^2, 2e, 1)$.

- a. $\frac{e^2}{2} - \frac{1}{e^2}$
- b. $\frac{e^4}{2} - \frac{e^2}{2} - 1$
- c. $\frac{e^2}{2} - \frac{1}{2} - \frac{1}{e^2}$
- d. $\frac{e^4}{2} + \frac{e^2}{2} - 1$
- e. $\frac{e^2}{2} + \frac{1}{2} - \frac{1}{e^2}$ Correct Choice

Solution: $f_{\text{ave}} = \frac{1}{L} \int_1^e f ds = \frac{1}{e^2} \int_1^e x |\vec{v}| dt = \frac{1}{e^2} \int_1^e t^2 \left(2t + \frac{1}{t}\right) dt = \frac{1}{e^2} \int_1^e (2t^3 + t) dt = \frac{1}{e^2} \left[\frac{t^4}{2} + \frac{t^2}{2}\right]_1^e$
 $= \left(\frac{e^2}{2} + \frac{1}{2}\right) - \frac{1}{e^2} \left(\frac{1}{2} + \frac{1}{2}\right) = \frac{e^2}{2} + \frac{1}{2} - \frac{1}{e^2}$

5. Find the plane tangent to $x \sin y + 2z \cos y = 2$ at $(x, y, z) = \left(\sqrt{2}, \frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$. The z -intercept is:

- a. 2
- b. $\sqrt{2}$ Correct Choice
- c. $\frac{1}{\sqrt{2}}$
- d. $\frac{1}{2\sqrt{2}}$
- e. 0

Solution: Let $f = x \sin y + 2z \cos y$ and $P = \left(\sqrt{2}, \frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$. Then

$\vec{\nabla} f = \langle \sin y, x \cos y - 2z \sin y, 2 \cos y \rangle$ and

$\vec{N} = \vec{\nabla} f|_P = \left\langle \frac{1}{\sqrt{2}}, \sqrt{2} \frac{1}{\sqrt{2}} - 2 \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}, 2 \frac{1}{\sqrt{2}} \right\rangle = \left\langle \frac{1}{\sqrt{2}}, 0, \sqrt{2} \right\rangle$.

So the tangent plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $\frac{1}{\sqrt{2}}x + \sqrt{2}z = \frac{1}{\sqrt{2}}\sqrt{2} + \sqrt{2} \frac{1}{\sqrt{2}} = 2$.

The z -intercept occurs when $x = y = 0$ or $\sqrt{2}z = 2$ or $z = \sqrt{2}$.

6. The velocity field of the water in a sink is $\vec{V} = \langle -x^2y, xy^2 \rangle$. Find the circulation of the water, $Circ = \oint \vec{V} \cdot d\vec{s}$, counterclockwise around the circle $x^2 + y^2 = 4$.
- 2π
 - $\frac{16}{3}\pi$
 - 8π Correct Choice
 - 32π
 - 64π

Solution: $Circ = \oint \vec{V} \cdot d\vec{s} = \oint P dx + Q dy$ where $P = -x^2y$ and $Q = xy^2$.

By Green's Theorem, $Circ = \iint (\partial_x Q - \partial_y P) dx dy = \iint (y^2 - -x^2) dx dy = \iint (x^2 + y^2) dx dy$

In polar coordinates, $Circ = \int_0^{2\pi} \int_0^2 r^2 r dr d\theta = 2\pi \left[\frac{r^4}{4} \right]_0^2 = 8\pi$

7. Compute $\int \vec{F} \cdot d\vec{s}$ for $\vec{F} = \langle 4x^3, 3y^2, 2z \rangle$ along the curve $\vec{r}(t) = \left(t \sin\left(\frac{\pi}{2}t\right), 2^t \cos\left(\frac{\pi}{2}t\right), 2^t \sin\left(\frac{\pi}{2}t\right) \right)$ from $t = 0$ to $t = 1$.
- 0
 - 1
 - 2
 - 3
 - 4 Correct Choice

Solution: A scalar potential is $f = x^4 + y^3 + z^2$.

The endpoints are $A = \vec{r}(0) = (0, 1, 0)$ and $B = \vec{r}(1) = (1, 0, 2)$.

By the FTCC, $\int \vec{F} \cdot d\vec{s} = \int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = (1^4 + 2^2) - (1^3) = 4$

8. Compute $\iint_S \vec{F} \cdot d\vec{S}$ for $\vec{F} = \langle 0, 0, -2 \rangle$ over the surface $\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, 16 - r^4)$ for $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$ oriented up and out.
HINT: Find a vector potential.
- -8π Correct Choice
 - -4π
 - -2π
 - 2π
 - 4π

Solution: We look for a vector potential with $A_3 = 0$. So we need to solve:

$$-\partial_z A_2 = F_1 = 0 \quad \partial_z A_1 = F_2 = 0 \quad \partial_x A_2 - \partial_y A_1 = -2$$

So $A_2 = f(x, y)$ $A_1 = g(x, y)$ $\partial_x f - \partial_y g = -2 \implies f = -x$ $g = y$ $\vec{A} = (y, -x, 0)$

The boundary curve ($r = 2$) is $\vec{r}(\theta) = (2 \cos \theta, 2 \sin \theta, 0)$. This correctly points counterclockwise.

The velocity is $\vec{v} = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle$. On the circle, $\vec{A} = \langle 2 \sin \theta, -2 \cos \theta, 0 \rangle$. By Stokes' Theorem:

$$\iint_S \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \oint_{\partial S} \vec{A} \cdot d\vec{s} = \int_0^{2\pi} \vec{A} \cdot \vec{v} d\theta = \int_0^{2\pi} -4 \sin^2 \theta - 4 \cos^2 \theta d\theta = -8\pi$$

9. Find the centroid of the solid hemisphere $0 \leq z \leq \sqrt{4 - x^2 - y^2}$.

- a. $(0, 0, \frac{1}{2})$
- b. $(0, 0, \frac{3}{4})$ Correct Choice
- c. $(0, 0, \frac{4}{3})$
- d. $(0, 0, 4\pi)$
- e. $(0, 0, 2\pi)$

Solution: By symmetry $\bar{x} = \bar{y} = 0$.

In spherical coordinates, $z = \rho \cos \varphi$ and $dV = \rho^2 \sin \varphi d\rho d\varphi d\theta$.

$$V_z = \iiint z dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho \cos \varphi \rho^2 \sin \varphi d\rho d\varphi d\theta = 2\pi \left[\frac{\sin^2 \varphi}{2} \right]_0^{\pi/2} \left[\frac{\rho^4}{4} \right]_0^2 = 4\pi$$

$$V = \frac{1}{2} \frac{4}{3} \pi 2^3 = \frac{16}{3} \pi \quad \bar{z} = \frac{V_z}{V} = \frac{3 \cdot 4\pi}{16\pi} = \frac{3}{4}$$

10. Find the area of the surface $\vec{R}(u, v) = (u, v, u + v)$ for $0 \leq u \leq 2$ and $0 \leq v \leq 3$.

- a. $12\sqrt{2}$
- b. $12\sqrt{3}$
- c. $6\sqrt{2}$
- d. $6\sqrt{3}$ Correct Choice
- e. $3\sqrt{2}$

Solution: $\vec{e}_u = \langle 1, 0, 1 \rangle$ $\vec{e}_v = \langle 0, 1, 1 \rangle$ $\vec{N} = \vec{e}_u \times \vec{e}_v = \langle -1, -1, 1 \rangle$ $|\vec{N}| = \sqrt{3}$

$$A = \iint |\vec{N}| du dv = \int_0^3 \int_0^2 \sqrt{3} du dv = 6\sqrt{3}$$

11. Find the flux of $\vec{F} = \langle y, -z, x \rangle$ upward thru the surface $\vec{R}(u, v) = (u, v, u + v)$ for $0 \leq u \leq 2$ and $0 \leq v \leq 3$.

- a. 30
- b. 24
- c. 18
- d. 12 Correct Choice
- e. 0

Solution: $\vec{N} = \vec{e}_u \times \vec{e}_v = \langle -1, -1, 1 \rangle$ which is correctly upward. $\vec{F} = \langle y, -z, x \rangle = \langle v, -u - v, u \rangle$

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N} du dv = \int_0^3 \int_0^2 (-v - (-u - v) + u) du dv = \int_0^3 \int_0^2 2u du dv = [u^2]_0^2 [v]_0^3 = 12$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 points) The Ideal Gas Law says the Pressure, P , Density, δ , and Temperature, T , are related by $P = k\delta T$ for some constant k . We will assume the atmosphere is an ideal gas with $k = 2$. A weather balloon measures the Density and Temperature to be

$$\delta = 0.1 \frac{\text{gm}}{\text{m}^3} \quad T = 270^\circ\text{K}$$

and their gradients to be

$$\vec{\nabla}\delta = \langle 0.01, 0.02, 0.03 \rangle \quad \vec{\nabla}T = \langle 3, 1, -1 \rangle$$

Find the gradient of the Pressure.

Solution: $\frac{\partial P}{\partial \delta} = kT = 2 \cdot 270 = 540$ $\frac{\partial P}{\partial T} = k\delta = 2 \cdot 0.1 = 0.2$

By the chain rule,

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial \delta} \frac{\partial \delta}{\partial x} + \frac{\partial P}{\partial T} \frac{\partial T}{\partial x} = 540 \cdot 0.01 + 0.2 \cdot 3 = 6$$

$$\frac{\partial P}{\partial y} = \frac{\partial P}{\partial \delta} \frac{\partial \delta}{\partial y} + \frac{\partial P}{\partial T} \frac{\partial T}{\partial y} = 540 \cdot 0.02 + 0.2 \cdot 1 = 11$$

$$\frac{\partial P}{\partial z} = \frac{\partial P}{\partial \delta} \frac{\partial \delta}{\partial z} + \frac{\partial P}{\partial T} \frac{\partial T}{\partial z} = 540 \cdot 0.03 + 0.2 \cdot (-1) = 16$$

So

$$\vec{\nabla}P = \langle 6, 11, 16 \rangle$$

13. (10 points) Find the largest value of $f = xy^2z^3$ on the plane $2x + 3y + 6z = 72$.

Solution 1: Lagrange Multipliers: Let $g = 2x + 3y + 6z$.

Then $\vec{\nabla}f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ and $\vec{\nabla}g = \langle 2, 3, 6 \rangle$.

The Lagrange equations are: $y^2z^3 = 2\lambda$, $2xyz^3 = 3\lambda$, $3xy^2z^2 = 6\lambda$

Then $xy^2z^3 = 2\lambda x = \frac{3}{2}\lambda y = 2\lambda z$ or $y = \frac{4}{3}x$ and $z = x$

From the constraint, $72 = 2x + 3y + 6z = 2x + 4x + 6x = 12x$.

So $x = 6$, $y = 8$, $z = 6$ and $f = xy^2z^3 = 6 \cdot 8^2 \cdot 6^3$

Solution 2: Eliminate $x = \frac{72 - 3y - 6z}{2} = 36 - \frac{3y}{2} - 3z$. Then

$$f = xy^2z^3 = \left(36 - \frac{3y}{2} - 3z\right)y^2z^3 = 36y^2z^3 - \frac{3}{2}y^3z^3 - 3y^2z^4$$

We find critical points:

$$f_y = 72yz^3 - \frac{9}{2}y^2z^3 - 6yz^4 = \frac{3}{2}yz^3(48 - 3y - 4z) = 0$$

$$f_z = 108y^2z^2 - \frac{9}{2}y^3z^2 - 12y^2z^3 = \frac{3}{2}y^2z^2(72 - 3y - 8z) = 0$$

Since $y \neq 0$ and $z \neq 0$, we need to solve $3y + 4z = 48$ and $3y + 8z = 72$

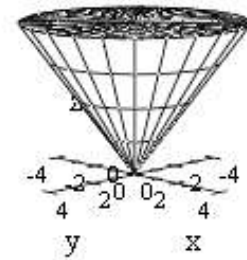
We subtract $4z = 24$ or $z = 6$

Substituting back gives: $3y + 24 = 48$ $3y = 24$ $y = 8$

and $x = 36 - \frac{3y}{2} - 3z = 36 - 12 - 18 = 6$. So $f = xy^2z^3 = 6 \cdot 8^2 \cdot 6^3$

14. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$

for the vector field $\vec{F} = (xz, yz, 3z\sqrt{x^2 + y^2})$ and the solid cone $\sqrt{x^2 + y^2} \leq z \leq 4$.



Be careful with orientations. Use the following steps:

First the Left Hand Side:

a. Compute the divergence:

$$\vec{\nabla} \cdot \vec{F} = z + z + 3\sqrt{x^2 + y^2}$$

b. Express the divergence and the volume element in the appropriate coordinate system:

$$\vec{\nabla} \cdot \vec{F} = 2z + 3r \quad dV = r \, dr \, d\theta \, dz$$

c. Compute the left hand side:

$$\begin{aligned} \iiint_V \vec{\nabla} \cdot \vec{F} \, dV &= \int_0^4 \int_0^{2\pi} \int_r^4 (2z + 3r) r \, dz \, d\theta \, dr = 2\pi \int_0^4 [z^2 + 3rz]_{z=r}^4 r \, dr \\ &= 2\pi \int_0^4 [(16 + 12r) - (r^2 + 3r^2)] r \, dr = 2\pi \int_0^4 (16r + 12r^2 - 4r^3) \, dr = 2\pi [8r^2 + 4r^3 - r^4]_0^4 \\ &= 2\pi(8 \cdot 4^2 + 4 \cdot 4^3 - 4^4) = 256\pi \end{aligned}$$

Second the Right Hand Side:

The boundary surface consists of a hemisphere H and a disk D with appropriate orientations.

d. Parametrize the disk D :

$$\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, 4)$$

e. Compute the tangent vectors:

$$\vec{e}_r = (\cos \theta, \sin \theta, 0)$$

$$\vec{e}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

f. Compute the normal vector:

$$\vec{N} = \hat{i}(0) - \hat{j}(0) + \hat{k}(r \cos^2 \theta - -r \sin^2 \theta) = (0, 0, r)$$

This is up as required.

g. Evaluate $\vec{F} = (xz, yz, 3z\sqrt{x^2 + y^2})$ on the disk:

$$\vec{F}|_{\vec{R}(r, \theta)} = (4r \cos \theta, 4r \sin \theta, 12r)$$

h. Compute the dot product:

$$\vec{F} \cdot \vec{N} = 12r^2$$

i. Compute the flux through D :

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^4 12r^2 \, dr \, d\theta = 2\pi [4r^3]_0^4 = 2\pi 4^3 = 512\pi$$

j. Parametrize the cone C :

$$\vec{R}(r, \theta) = (r \cos \theta, r \sin \theta, r)$$

k. Compute the tangent vectors:

$$\vec{e}_r = (\cos \theta, \sin \theta, 1)$$

$$\vec{e}_\theta = (-r \sin \theta, r \cos \theta, 0)$$

l. Compute the normal vector:

$$\begin{aligned}\vec{N} &= \hat{i}(-r \cos \theta) - \hat{j}(r \sin \theta) + \hat{k}(r \cos^2 \theta + r \sin^2 \theta) \\ &= (-r \cos \theta, -r \sin \theta, r)\end{aligned}$$

This is oriented up and in. We need down and out. Reverse it:

$$\vec{N} = (r \cos \theta, r \sin \theta, -r)$$

m. Evaluate $\vec{F} = (xz, yz, 3z\sqrt{x^2 + y^2})$ on the cone:

$$\vec{F}|_{\vec{R}(r, \theta)} = (r^2 \cos \theta, r^2 \sin \theta, 3r^2)$$

n. Compute the dot product:

$$\vec{F} \cdot \vec{N} = r^3 \cos^2 \theta + r^3 \sin^2 \theta - 3r^3 = -2r^3$$

o. Compute the flux through C :

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^4 -2r^3 dr d\theta = -2\pi \left[\frac{r^4}{2} \right]_0^4 = -4^4 \pi = -256\pi$$

p. Compute the **TOTAL** right hand side:

$$\iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} + \iint_C \vec{F} \cdot d\vec{S} = 512\pi - 256\pi = 256\pi \quad \text{which agrees with (c).}$$