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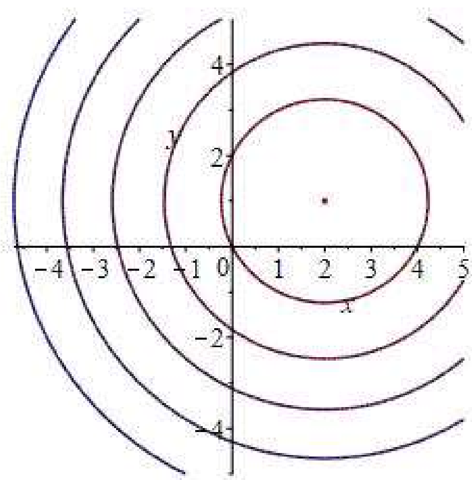
MATH 251 Exam 2 Version B Fall 2018

Sections 504/505 Solutions P. Yasskin

1-11	/55	13	/25
12	/20	EC	/5
		Total	/105

Multiple Choice: (5 points each. No part credit.)

1. Which of these functions has the contour plot at the right?



a. $\sqrt{x^2+y^2-4x-2y+9}$ Correct Choice

b. $\sqrt{x^2+y^2+4x+2y+5}$

c. $\sqrt{x^2+y^2+4x-2y+1}$

d. $x^2+y^2-4x+2y$

e. $x^2+y^2+4x+2y$

Solution: Completing the square, the functions are

a. $\sqrt{(x-2)^2+(y-1)^2+4}$ level sets are all circles centered at $(2,1)$ Correct

b. $\sqrt{(x+2)^2+(y+1)^2}$ level sets are all circles centered at $(-2,-1)$

c. $\sqrt{(x+2)^2+(y-1)^2-4}$ The level sets are circles centered at $(-2,1)$ of radius ≥ 2

d. $(x-2)^2+(y+1)^2-5$ level sets are all circles centered at $(2,-1)$

e. $(x+2)^2+(y+1)^2-5$ level sets are all circles centered at $(-2,-1)$

2. If $f = x\cos y - y\sin x$ which of the following is INCORRECT?

a. $\dfrac{\partial^3 f}{\partial x\partial x\partial x} = y\cos x$

b. $\dfrac{\partial^3 f}{\partial y\partial x\partial x} = \sin x$

c. $\dfrac{\partial^3 f}{\partial x\partial y\partial x} = -\sin x$ Correct Choice

d. $\dfrac{\partial^3 f}{\partial x\partial x\partial y} = \sin x$

e. $\dfrac{\partial^3 f}{\partial y\partial y\partial y} = x\sin y$

Solution: By Clairaut's Theorem, $f_{yxx} = f_{xyx} = f_{xxy}$. Since answer (b) equals answer (d) but not answer (c), answer (c) must be wrong.

3. The partial derivative $\left.\frac{\partial f}{\partial x}\right|_{(2,3)}$ gives the

a. slope at $y = 3$ of the x-trace of f with x fixed at 2.

b. slope at $x = 2$ of the x-trace of f with y fixed at 3. Correct Choice

c. slope at $y = 3$ of the y-trace of f with x fixed at 2.

d. slope at $x = 2$ of the y-trace of f with y fixed at 3.

Solution: An x-trace has y fixed while a y-trace has x fixed.
$\left.\frac{\partial f}{\partial x}\right|_{(2,3)}$ needs y fixed at 3 while we differentiate with respect to x at $x = 2$.

4. Find the tangent plane to the graph of $z = x^2y^3$ at $(x,y) = (2,1)$. The z-intercept is

a. 20

b. 16

c. 4

d. -16 Correct Choice

e. -20

Solution: Let $f(x,y) = x^2y^3$. The tangent plane is $z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$.
We compute $f(2,1) = 4$ $f_x(x,y) = 2xy^3$ $f_x(2,1) = 4$ $f_y(x,y) = 3x^2y^2$ $f_y(2,1) = 12$
So the tangent plane is $z = 4 + 4(x-2) + 12(y-1) = 4x + 12y - 16$ and the z-intercept is -16.

5. The equation $x^3z^3 - y^2z^2 = -1$ implicitly defines z as a function of x and y.
Find $\frac{\partial z}{\partial y}$ at $(x,y,z) = (2,3,1)$.

a. 2

b. 1 Correct Choice

c. 0

d. -1

e. -2

Solution: $x^3 3z^2 \frac{\partial z}{\partial y} - 2yz^2 - y^2 2z \frac{\partial z}{\partial y} = 0$ $24\frac{\partial z}{\partial y} - 6 - 18\frac{\partial z}{\partial y} = 0$ $\frac{\partial z}{\partial y} = 1$

6. Find the equation of the plane tangent to the surface $x^3z^3 - y^2z^2 = -1$ at $(x,y,z) = (2,3,1)$.
The z-intercept is

a. $c = -12$

b. $c = -2$

c. $c = 2$ Correct Choice

d. $c = 6$

e. $c = 12$

Solution: Let $F = x^3z^3 - y^2z^2$. Then $\vec{\nabla}F = \langle 3x^2z^3, -2yz^2, 3x^3z^2 - 2y^2z \rangle$.
So the normal is $\vec{N} = \left.\vec{\nabla}F\right|_{(2,3,1)} = \langle 12, -6, 6 \rangle$ and the tangent plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or
$12x - 6y + 6z = 12 \cdot 2 - 6 \cdot 3 + 6 \cdot 1 = 12$. So the z-intercept is $c = 2$.

7. The strength, S, of a support beam of length L, width W and height H is given by $S = \frac{WH^2}{L}$. Currently, $L = 50$ cm, $W = 5$ cm and $H = 10$ cm. Use the linear approximation to estimate the change in the strength if L increases by 5 cm, W increases by 0.5 cm and H increases by 1 cm?

a. 10

b. 8

c. 6

d. 4

e. 2 Correct Choice

Solution: The partial derivatives of S are:

$$\frac{\partial S}{\partial L} = -\frac{WH^2}{L^2} = -\frac{5 \cdot 10^2}{50^2} = -\frac{1}{5} \qquad \frac{\partial S}{\partial W} = \frac{H^2}{L} = \frac{10^2}{50} = 2 \qquad \frac{\partial S}{\partial H} = \frac{2WH}{L} = \frac{2 \cdot 5 \cdot 10}{50} = 2$$

The change in strength is approximately its differential:

$$dS = \frac{\partial S}{\partial L}dL + \frac{\partial S}{\partial W}dW + \frac{\partial S}{\partial H}dH = -\frac{1}{5} \cdot 5 + 2 \cdot 0.5 + 2 \cdot 1 = 2$$

8. Dark Invader is flying through a dark matter field whose density is given by $\delta = xyz^2$. If Dark's current position is $\vec{r}(2) = \langle 3,2,1 \rangle$ and his velocity is $\vec{v}(2) = \langle 1,2,1 \rangle$, find the rate at which the density of dark matter is changing as seen by Dark.

a. 10

b. $10\sqrt{6}$

c. $20\sqrt{6}$

d. 20 Correct Choice

e. $\frac{20}{\sqrt{6}}$

Solution: $\vec{\nabla}\delta = \langle yz^2, xz^2, 2xyz \rangle$ $\quad \vec{\nabla}\delta\Big|_{(3,2,1)} = \langle 2,3,12 \rangle$ $\quad \frac{d\delta}{dt} = \vec{v} \cdot \vec{\nabla}\delta = 2 + 6 + 12 = 20$

9. When there is no wind, a weather balloon floats in the direction of **decreasing** air density. If the air density is $\delta = x^2 + y^2 + z^3$ and the balloon is located at $(x,y,z) = (2,6,1)$, find the vector direction in which the balloon floats.

a. $\left\langle \frac{4}{13}, \frac{12}{13}, \frac{3}{13} \right\rangle$

b. $\left\langle \frac{-4}{13}, \frac{12}{13}, \frac{-3}{13} \right\rangle$

c. $\left\langle \frac{-4}{13}, \frac{-12}{13}, \frac{-3}{13} \right\rangle$ Correct Choice

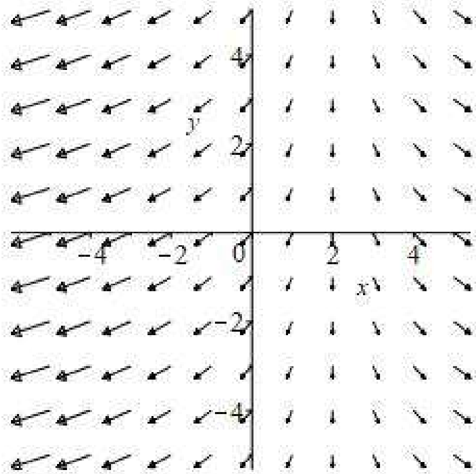
d. $\left\langle \frac{4}{13}, \frac{-12}{13}, \frac{3}{13} \right\rangle$

Solution: $\vec{\nabla}\delta = \langle 2x, 2y, 3z^2 \rangle$ $\quad \vec{\nabla}\delta\Big|_{(2,6,1)} = \langle 4,12,3 \rangle$ $\quad \left|\vec{\nabla}\delta\right| = \sqrt{16 + 144 + 9} = 13$

The density **decreases** in the direction $\hat{u} = \frac{-\vec{\nabla}\delta}{\left|\vec{\nabla}\delta\right|} = \left\langle \frac{-4}{13}, \frac{-12}{13}, \frac{-3}{13} \right\rangle$.

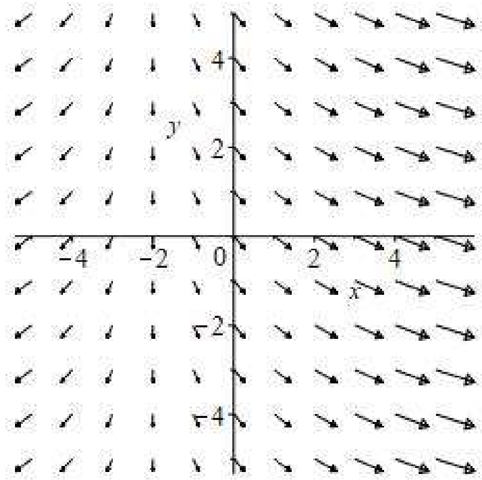
10. Which is the plot of the vector field $\vec{F} = \langle x-2, -2 \rangle$?

a.

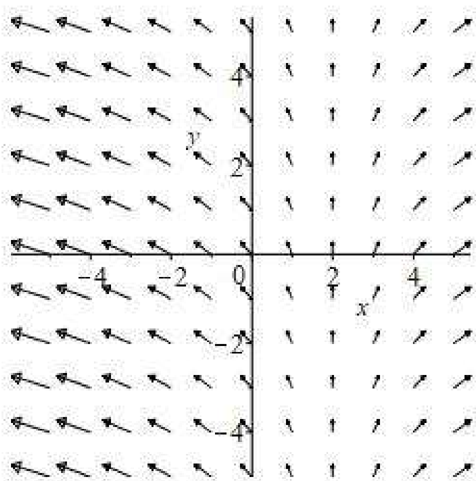


Correct Choice

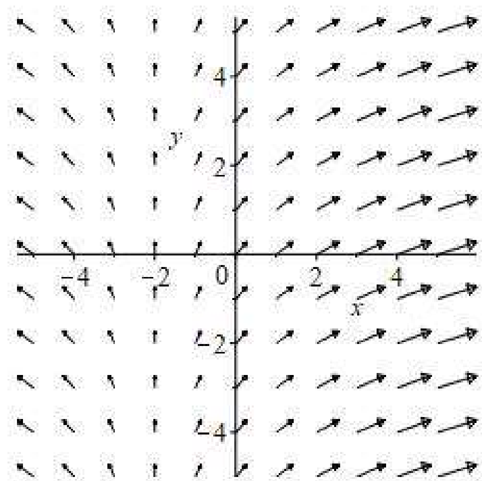
c.



b.



d.



Solution: The y-coordinate is -2. So the vector field points down. The x-coordinate is $x-2$. So the x coordinate is 0 when $x = 2$. That is plot (a).

11. Find a scalar potential, $f(x,y,z)$, for $\vec{F} = \left\langle -\frac{yz}{x^2}, \frac{z}{x}, \frac{y}{x} \right\rangle$. Then $f(4,4,4) - f(1,1,1) =$

a. 1

b. 2

c. 3 Correct Choice

d. 4

e. 5

Solution: $\vec{\nabla}f - \vec{F}$ $\quad \partial_x f = -\frac{yz}{x^2}, \quad \partial_y f = \frac{z}{x}, \quad \partial_z f = \frac{y}{x} \quad f = \frac{yz}{x} \quad f(4,4,4) - f(1,1,1) = 4 - 1 = 3$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (20 points) Find the point(s), $X = (x,y,z)$, on the hyperboloid $x^2 + y^2 - z^2 = 1$ where the normal vector points in the same direction as $\vec{v} = \langle 1,3,-3 \rangle$.

Solution: Let $F = x^2 + y^2 - z^2$. Then the normal is $\vec{N} = \vec{\nabla} F = \langle 2x, 2y, -2z \rangle$. The normal is in the same direction as $\vec{v}$ if $\vec{N} = \lambda \vec{v}$, i.e.

$$2x = \lambda \qquad 2y = \lambda 3 \qquad -2z = -\lambda 3$$

Since $\lambda = 2x$, we have $2y = 6x$ and $-2z = -6x$ or $y = 3x$ and $z = 3x$. We substitute these into the equation of the hyperboloid:

$$1 = x^2 + y^2 - z^2 = x^2 + 9x^2 - 9x^2 = x^2 \qquad \Rightarrow \qquad x = \pm 1$$

So the points are

$$(x,y,z) = (1,3,3) \qquad \text{and} \qquad (x,y,z) = (-1,-3,-3)$$

13. (25 points+5 points extra credit) Find the point, $X = (x, y, z)$, on the upper half of the hyperboloid $x^2 + y^2 - z^2 = 1$ which is closest to the point $P = (6, 8, 0)$. What is the distance?

You may solve by either method. There is 5 points extra credit for solving by both methods.

Solution: We are minimizing the distance from P to X, or the square of the distance:

$$f = (x-6)^2 + (y-8)^2 + z^2$$

subject to the constraint:

$$g = x^2 + y^2 - z^2 = 1$$

....

Method: **Lagrange Multipliers**:
We compute the gradients:

$$\vec{\nabla} f = \langle 2(x-6), 2(y-8), 2z \rangle \qquad \vec{\nabla} g = \langle 2x, 2y, -2z \rangle$$

So the Lagrange equations are:

$$2(x-6) = 2x\lambda \qquad 2(y-8) = 2y\lambda \qquad 2z = -2z\lambda$$

The z equation gives $\lambda = -1$. Then the x and y equations give:

$$x - 6 = -x \qquad y - 8 = -y$$

or $x = 3$ and $y = 4$. Substituting into the constraint gives

$$z^2 = x^2 + y^2 - 1 = 9 + 16 - 1 = 24 \qquad z = \sqrt{24} = 2\sqrt{6}$$

....

Method: **Eliminate the Constraint**:
We solve the constraint for $z^2 = x^2 + y^2 - 1$ and substitute into the square of the distance:

$$f = (x-6)^2 + (y-8)^2 + x^2 + y^2 - 1$$

We set the derivatives equal to 0 and solve:

$$f_x = 2(x-6) + 2x = 0 \qquad \Rightarrow \qquad x = 3$$

$$f_y = 2(y-8) + 2y = 0 \qquad \Rightarrow \qquad y = 4$$

We substitute back to get

$$z = \sqrt{x^2 + y^2 - 1} = \sqrt{9 + 16 - 1} = \sqrt{24} = 2\sqrt{6}$$

....

The point is $\left(3, 4, 2\sqrt{6}\right)$ and the distance is

$$D = \sqrt{f} = \sqrt{(x-6)^2 + (y-4)^2 + z^2} = \sqrt{9 + 16 + 24} = 7$$