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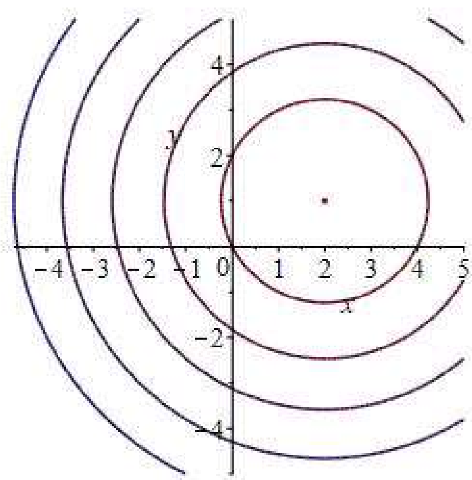
MATH 251 Exam 2 Version H Fall 2018

Sections 200/202 Solutions P. Yasskin

1-11	/55	13	/25
12	/20	EC	/5
		Total	/105

Multiple Choice: (5 points each. No part credit.)

1. Which of these functions has the contour plot at the right?



a. $\sqrt{x^2+y^2+4x-2y+1}$

b. $\sqrt{x^2+y^2+4x+2y+5}$

c. $\sqrt{x^2+y^2-4x-2y+9}$ Correct Choice

d. $x^2+y^2-4x+2y$

e. $x^2+y^2+4x+2y$

Solution: Completing the square, the functions are

a. $\sqrt{(x+2)^2+(y-1)^2-4}$ The level sets are circles centered at $(-2,1)$ of radius ≥ 2

b. $\sqrt{(x+2)^2+(y+1)^2}$ level sets are all circles centered at $(-2,-1)$

c. $\sqrt{(x-2)^2+(y-1)^2+4}$ level sets are all circles centered at $(2,1)$ Correct

d. $(x-2)^2+(y+1)^2-5$ level sets are all circles centered at $(2,-1)$

e. $(x+2)^2+(y+1)^2-5$ level sets are all circles centered at $(-2,-1)$

2. If $\vec{F} = \langle 2xyz, -3y^2z, 2yz^2\rangle$, which of the following is FALSE?

a. $\vec{\nabla}\cdot\vec{F} = 0$

b. $\vec{\nabla}\times\vec{F} = \langle 2z^2+3y^2, 2xy, -2xz\rangle$

c. $\vec{F}$ has a vector potential.

d. $\vec{F}$ has a scalar potential. Correct Choice

Solution: $\vec{\nabla}\cdot\vec{F} = 2yz - 6yz + 4yz = 0$

$$\vec{\nabla}\times\vec{F} = \left|\begin{pmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2xyz & -3y^2z & 2yz^2 \end{pmatrix}\right| = \hat{\imath}(2z^2 - -3y^2) - \hat{\jmath}(0-2xy) + \hat{k}(0-2xz) = \langle 2z^2+3y^2, 2xy, -2xz\rangle$$

$\vec{F}$ probably has a vector potential since $\vec{\nabla}\cdot\vec{F} = 0$.

$\vec{F}$ cannot have a scalar potential since $\vec{\nabla}\times\vec{F} = 0$.

3. The partial derivative $\left.\frac{\partial f}{\partial x}\right|_{(2,3)}$ gives the

a. slope at $y = 3$ of the x-trace of f with x fixed at 2.
b. slope at $x = 2$ of the x-trace of f with y fixed at 3. Correct Choice
c. slope at $y = 3$ of the y-trace of f with x fixed at 2.
d. slope at $x = 2$ of the y-trace of f with y fixed at 3.

Solution: An x-trace has y fixed while a y-trace has x fixed.
$\left.\frac{\partial f}{\partial x}\right|_{(2,3)}$ needs y fixed at 3 while we differentiate with respect to x at $x = 2$.

4. Find the tangent plane to the graph of $z = x^2y^3$ at $(x,y) = (2,1)$. The z-intercept is

a. -16 Correct Choice
b. 16
c. 4
d. -20
e. 20

Solution: Let $f(x,y) = x^2y^3$. The tangent plane is $z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$.
We compute $f(2,1) = 4$ $f_x(x,y) = 2xy^3$ $f_x(2,1) = 4$ $f_y(x,y) = 3x^2y^2$ $f_y(2,1) = 12$
So the tangent plane is $z = 4 + 4(x-2) + 12(y-1) = 4x + 12y - 16$ and the z-intercept is -16.

5. The equation $x^3z^3 - y^2z^2 = -1$ implicitly defines z as a function of x and y.
Find $\frac{\partial z}{\partial x}$ at $(x,y,z) = (2,3,1)$.

a. 2
b. 1
c. 0
d. -1
e. -2 Correct Choice

Solution: $3x^2z^3 + x^3 3z^2\frac{\partial z}{\partial x} - y^2 2z\frac{\partial z}{\partial x} = 0$ $\quad 12 + 24\frac{\partial z}{\partial x} - 18\frac{\partial z}{\partial x} = 0$ $\quad \frac{\partial z}{\partial x} = -2$

6. Find the equation of the line perpendicular to the surface $x^3z^3 - y^2z^2 = -1$ at $(x,y,z) = (2,3,1)$.
It intersects the xy-plane at

a. $(0,4,0)$ Correct Choice
b. $(-2,5,0)$
c. $(-4,6,0)$
d. $(4,2,0)$
e. $(8,0,0)$

Solution: Let $F = x^3z^3 - y^2z^2$. Then $\vec{\nabla}F = \langle 3x^2z^3, -2yz^2, 3x^3z^2 - 2y^2z\rangle$
So the normal is $\vec{N} = \left.\vec{\nabla}F\right|_{(2,3,1)} = \langle 12,-6,6\rangle$ and the normal line is $X = P + t\vec{N}$ or
$(x,y,z) = (2+12t, 3-6t, 1+6t)$. It intersects the xy-plane when $z = 0$ or $1 + 6t = 0$ or $t = \frac{-1}{6}$.
So it intersects the xy-plane at $(x,y,z) = \left(2 + 12\left(\frac{-1}{6}\right), 3 - 6\left(\frac{-1}{6}\right), 1 + 6\left(\frac{-1}{6}\right)\right) = (0,4,0)$

7. The strength, S, of a support beam of length L, width W and height H is given by $S = \frac{WH^2}{L}$. Currently, $L = 50$ cm, $W = 5$ cm and $H = 10$ cm. Use the linear approximation to estimate the change in the strength if L increases by 5 cm, W increases by 0.5 cm and H increases by 2 cm.

a. 2

b. 4 Correct Choice

c. 6

d. 8

e. 10

Solution: The partial derivatives of S are:

$$\frac{\partial S}{\partial L} = -\frac{WH^2}{L^2} = -\frac{5 \cdot 10^2}{50^2} = -\frac{1}{5} \qquad \frac{\partial S}{\partial W} = \frac{H^2}{L} = \frac{10^2}{50} = 2 \qquad \frac{\partial S}{\partial H} = \frac{2WH}{L} = \frac{2 \cdot 5 \cdot 10}{50} = 2$$

The change in strength is approximately its differential:

$$dS = \frac{\partial S}{\partial L}dL + \frac{\partial S}{\partial W}dW + \frac{\partial S}{\partial H}dH = -\frac{1}{5} \cdot 5 + 2 \cdot 0.5 + 2 \cdot 2 = 4$$

8. Dark Invader is flying through a dark matter field whose density is given by $\delta = xyz^2$. If Dark's current position is $\vec{r}(2) = \langle 3,2,1 \rangle$ and his velocity is $\vec{v}(2) = \langle 1,2,1 \rangle$, find the rate at which the density of dark matter is changing as seen by Dark.

a. 10

b. $10\sqrt{6}$

c. $20\sqrt{6}$

d. 20 Correct Choice

e. $\frac{20}{\sqrt{6}}$

Solution: $\vec{\nabla}\delta = \langle yz^2, xz^2, 2xyz \rangle \qquad \vec{\nabla}\delta\Big|_{(3,2,1)} = \langle 2,3,12 \rangle \qquad \frac{d\delta}{dt} = \vec{v} \cdot \vec{\nabla}\delta = 2 + 6 + 12 = 20$

9. When there is no wind, a weather balloon floats in the direction of **decreasing** air density. If the air density is $\delta = x^2 + y^2 + z^3$ and the balloon is located at $(x,y,z) = (2,6,1)$, find the vector direction in which the balloon floats.

a. $\left\langle \frac{4}{13}, \frac{12}{13}, \frac{3}{13} \right\rangle$

b. $\left\langle \frac{-4}{13}, \frac{12}{13}, \frac{-3}{13} \right\rangle$

c. $\left\langle \frac{-4}{13}, \frac{-12}{13}, \frac{-3}{13} \right\rangle$ Correct Choice

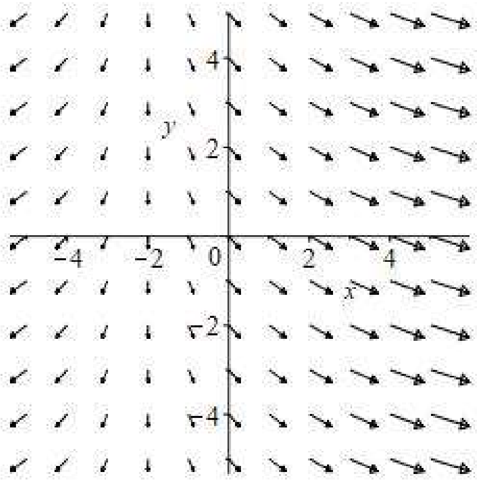
d. $\left\langle \frac{4}{13}, \frac{-12}{13}, \frac{3}{13} \right\rangle$

Solution: $\vec{\nabla}\delta = \langle 2x, 2y, 3z^2 \rangle \qquad \vec{\nabla}\delta\Big|_{(2,6,1)} = \langle 4,12,3 \rangle \qquad \left|\vec{\nabla}\delta\right| = \sqrt{16 + 144 + 9} = 13$

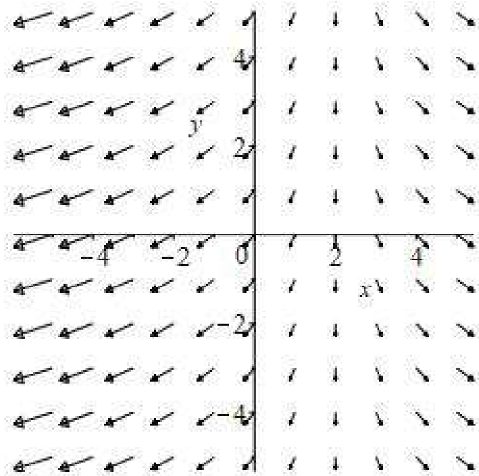
The density **decreases** in the direction $\hat{u} = \frac{-\vec{\nabla}\delta}{\left|\vec{\nabla}\delta\right|} = \left\langle \frac{-4}{13}, \frac{-12}{13}, \frac{-3}{13} \right\rangle$.

10. Which is the plot of the vector field $\vec{F} = \langle x-2, 2\rangle$?

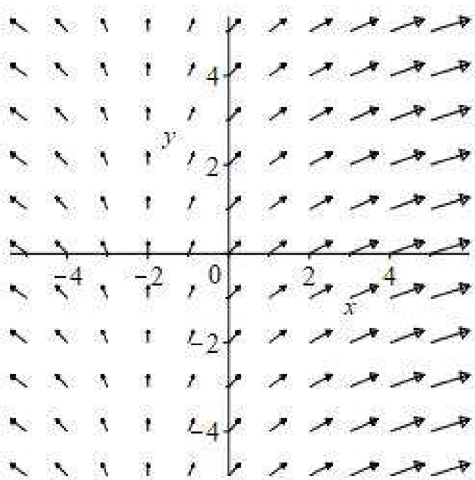
a.



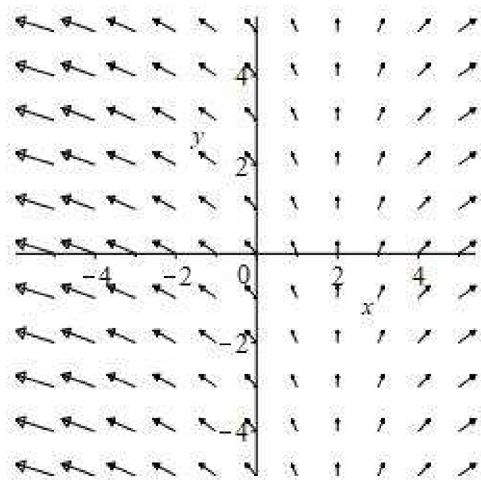
c.



b.



d.



Correct Choice

Solution: The y-coordinate is 2. So the vector field points up. The x–coordinate is $x-2$. So the x coordinate is 0 when $x=2$. That is plot (d).

11. Find a scalar potential, $f(x,y,z)$, for $\vec{F} = \left\langle -\frac{yz}{x^2}, \frac{z}{x}, \frac{y}{x} \right\rangle$. Then $f(2,2,2) - f(1,1,1) =$

a. 5

b. 4

c. 3

d. 2

e. 1 Correct Choice

Solution: $\vec{\nabla} f - \vec{F}$ $\quad \partial_x f = -\frac{yz}{x^2}, \quad \partial_y f = \frac{z}{x}, \quad \partial_z f = \frac{y}{x} \quad f = \frac{yz}{x} \quad f(2,2,2) - f(1,1,1) = 2 - 1 = 1$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 points) Find the point(s), $X = (x,y,z)$, on the hyperboloid $x^2 + y^2 - z^2 = 1$ where the normal vector points in the same direction as $\vec{v} = \langle 1, 5, -5 \rangle$.

Solution: Let $F = x^2 + y^2 - z^2$. Then the normal is $\vec{N} = \vec{\nabla} F = \langle 2x, 2y, -2z \rangle$. The normal is in the same direction as $\vec{v}$ if $\vec{N} = \lambda \vec{v}$, i.e.

$$2x = \lambda \qquad 2y = \lambda 5 \qquad -2z = -\lambda 5$$

Since $\lambda = 2x$, we have $2y = 10x$ and $-2z = -10x$ or $y = 5x$ and $z = 5x$. We substitute these into the equation of the hyperboloid:

$$1 = x^2 + y^2 - z^2 = x^2 + 25x^2 - 25x^2 = x^2 \qquad \Rightarrow \qquad x = \pm 1$$

So the points are

$$(x,y,z) = (1,5,5) \qquad \text{and} \qquad (x,y,z) = (-1,-5,-5)$$

13. (15 points+5 points extra credit) Find the point, $X = (x,y,z)$, on the upper half of the hyperboloid $x^2 + y^2 - z^2 = 1$ which is closest to the point $P = (4,6,0)$. What is the distance?

You may solve by either method. There is 5 points extra credit for solving by both methods.

Solution: We are minimizing the distance from P to X, or the square of the distance:

$$f = (x-4)^2 + (y-6)^2 + z^2$$

subject to the constraint:

$$g = x^2 + y^2 - z^2 = 1$$

....

Method: **Lagrange Multipliers**:

We compute the gradients:

$$\vec{\nabla} f = \langle 2(x-4), 2(y-6), 2z \rangle \qquad \vec{\nabla} g = \langle 2x, 2y, -2z \rangle$$

So the Lagrange equations are:

$$2(x-4) = 2x\lambda \qquad 2(y-6) = 2y\lambda \qquad 2z = -2z\lambda$$

The z equation gives $\lambda = -1$. Then the x and y equations give:

$$x - 4 = -x \qquad y - 6 = -y$$

or $x = 2$ and $y = 3$. Substituting into the constraint gives

$$z^2 = x^2 + y^2 - 1 = 4 + 9 - 1 = 12 \qquad z = \sqrt{12} = 2\sqrt{3}$$

....

Method: **Eliminate the Constraint**:

We solve the constraint for $z^2 = x^2 + y^2 - 1$ and substitute into the square of the distance:

$$f = (x-4)^2 + (y-6)^2 + x^2 + y^2 - 1$$

We set the derivatives equal to 0 and solve:

$$f_x = 2(x-4) + 2x = 0 \qquad \Rightarrow \qquad x = 2$$

$$f_y = 2(y-6) + 2y = 0 \qquad \Rightarrow \qquad y = 3$$

We substitute back to get

$$z = \sqrt{x^2 + y^2 - 1} = \sqrt{4 + 9 - 1} = \sqrt{12} = 2\sqrt{3}$$

....

The point is $\left(2, 3, 2\sqrt{3}\right)$ and the distance is

$$D = \sqrt{f} = \sqrt{(x-4)^2 + (y-6)^2 + z^2} = \sqrt{4 + 9 + 12} = 5$$