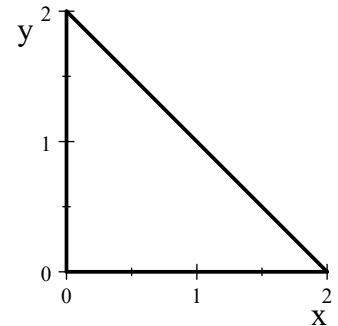


Name _____

MATH 251 Final Version A Fall 2018
 Sections 504 Solutions P. Yasskin
 Multiple Choice: (6 points each. No part credit.)

1-10	/60	12	/20
11	/5	13	/25
		Total	/110

1. Compute $I = \int_{\partial R} (2y + 3x^2y^2) dx + (5x + 2x^3y) dy$ over the complete boundary of the triangle shown at the right traversed counterclockwise.
 HINT: Use a theorem.



- a. $I = 3$
- b. $I = 6$ Correct Choice
- c. $I = 7$
- d. $I = 12$
- e. $I = 14$

Solution: Green's Theorem says: $\int_{\partial R} P dx + Q dy = \iint_R (\partial_x Q - \partial_y P) dA$.

We identify: $P = 2y + 3x^2y^2$ and $Q = 5x + 2x^3y$. So $\partial_x Q - \partial_y P = (5 + 6x^2y) - (2 + 6x^2y) = 3$.

Consequently: $I = \iint_R 3 dA = 3Area = 3 \cdot \frac{1}{2} \cdot b \cdot h = 3 \cdot \frac{1}{2} \cdot 2 \cdot 2 = 6$

2. Compute $\int_{(1,1,1)}^{(8,4,2)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = \langle y^2z^2, 2xyz^2, 2xy^2z \rangle$ along the curve $\vec{r}(t) = \langle t^3, t^2, t \rangle$.

HINT: Find a scalar potential.

- a. -512
- b. -511
- c. 0
- d. 511 Correct Choice
- e. 512

Solution: By inspection, a scalar potential is $f = xy^2z^2$ since $\vec{\nabla}f = \langle y^2z^2, 2xyz^2, 2xy^2z \rangle = \vec{F}$.
 By the Fundamental Theorem of Calculus for Curves,

$$\int_{(1,1,1)}^{(8,4,2)} \vec{F} \cdot d\vec{s} = \int_{(1,1,1)}^{(8,4,2)} \vec{\nabla}f \cdot d\vec{s} = f(8,4,2) - f(1,1,1) = 8 \cdot 4^2 \cdot 2^2 - 1 \cdot 1^2 \cdot 1^2 = 511$$

3. Compute $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the quartic surface $z = x^4 + 2x^2y^2 + y^4$ with $z \leq 16$ oriented **up** and **in**, for $\vec{F} = \langle -yz, xz, z^2 \rangle$.

HINT: Use a theorem.

- a. 128π Correct Choice
- b. 256π
- c. 512π
- d. 1024π
- e. 2048π



Solution: Stokes' Theorem says $\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$. In cylindrical coordinates, the surface can be written as $z = (x^2 + y^2)^2 = r^4$. So its boundary is $z = r^4 = 16$ or $r = 2$. This is a circle which may be parametrized as $\vec{r}(\theta) = \langle 2 \cos \theta, 2 \sin \theta, 16 \rangle$. Then $\vec{v} = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle$ which is correctly ccw. On the circle, $\vec{F} = \langle -32 \sin \theta, 32 \cos \theta, 16^2 \rangle$. So $\vec{F} \cdot \vec{v} = 64 \sin^2 \theta + 64 \cos^2 \theta + 0 = 64$. Then

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v} d\theta = \int_0^{2\pi} 64 d\theta = 128\pi$$

4. The two legs of a right triangle are \vec{a} and \vec{b} and the hypotenuse is \vec{c} . So $\vec{a} \perp \vec{b}$ and $\vec{c} = \vec{a} + \vec{b}$. Given that $\vec{c} = \langle 9, 9, -9 \rangle$ and the direction of \vec{a} is $\hat{a} = \langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \rangle$, find the magnitude $|\vec{b}|$.
- a. $|\vec{b}| = 36$
 - b. $|\vec{b}| = 18\sqrt{2}$
 - c. $|\vec{b}| = 18$
 - d. $|\vec{b}| = 9\sqrt{2}$ Correct Choice
 - e. $|\vec{b}| = 9$

Solution: \vec{a} is the projection of \vec{c} onto \hat{a} . Since $\vec{c} \cdot \hat{a} = 6 - 3 + 6 = 9$ and $|\hat{a}| = 1$, we have:

$$\vec{a} = \text{proj}_{\hat{a}} \vec{c} = \frac{\vec{c} \cdot \hat{a}}{|\hat{a}|^2} \hat{a} = \frac{9}{1} \left\langle \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3} \right\rangle = \langle 6, -3, -6 \rangle$$

Then $\vec{b} = \vec{c} - \vec{a} = \langle 9, 9, -9 \rangle - \langle 6, -3, -6 \rangle = \langle 3, 12, -3 \rangle$. So $|\vec{b}| = \sqrt{9 + 144 + 9} = 9\sqrt{2}$.

5. An ant is walking across a frying pan where the temperature is $T = \frac{1}{12}x^3y^2$. If the ant is currently at $P = (2, 3)$, in what unit vector direction should the ant walk to reduce the temperature as fast as possible?

- a. $\left\langle \frac{9}{13}, \frac{4}{13} \right\rangle$
- b. $\left\langle \frac{-9}{5}, \frac{-4}{5} \right\rangle$
- c. $\left\langle \frac{9}{5}, \frac{4}{5} \right\rangle$
- d. $\left\langle \frac{9}{\sqrt{97}}, \frac{4}{\sqrt{97}} \right\rangle$
- e. $\left\langle \frac{-9}{\sqrt{97}}, \frac{-4}{\sqrt{97}} \right\rangle$ Correct Choice

Solution: $\vec{\nabla}T = \left\langle \frac{1}{4}x^2y^2, \frac{1}{6}x^3y \right\rangle = \langle 9, 4 \rangle \quad \vec{u} = -\vec{\nabla}T = \langle -9, -4 \rangle$

$|\vec{u}| = \sqrt{9^2 + 4^2} = \sqrt{97} \quad \hat{u} = \frac{\vec{u}}{|\vec{u}|} = \left\langle \frac{-9}{\sqrt{97}}, \frac{-4}{\sqrt{97}} \right\rangle$

6. The point $(1, 2)$ is a critical point of the function $f(x, y) = 16x^4 + y^4 - 32xy$. Classify the point $(1, 2)$ using the Second Derivative Test.

- a. Local Minimum Correct Choice
- b. Local Maximum
- c. Saddle Point
- d. Inflection Point
- e. Test Fails

Solution: $f_x = 64x^3 - 32y \quad f_x(1, 2) = 64 - 64 = 0 \quad f_y = 4y^3 - 32x \quad f_y(1, 2) = 32 - 32 = 0$

$f_{xx} = 192x^2 \quad f_{xx}(1, 2) = 192 > 0 \quad f_{yy} = 12y^2 \quad f_{yy}(1, 2) = 48 \quad f_{xy} = -32 \quad f_{xy}(1, 2) = -32$

$D = f_{xx}f_{yy} - f_{xy}^2 \quad D(1, 2) = 192 \cdot 48 - 32^2 = 8192 > 0 \quad \text{Local Minimum}$

7. Find the mass of the piece of the solid paraboloid $z = x^2 + y^2$ for $2 \leq z \leq 4$ if the density is $\delta = z$.

- a. 64π
- b. 60π
- c. $\frac{112}{3}\pi$
- d. $\frac{56}{3}\pi$ Correct Choice
- e. 20π



Solution: In cylindrical coordinates, $dV = r dr d\theta dz$ and the paraboloid is $z = r^2$. Since z goes between constant limits, we put the z integral outside and write the cone as $r = \sqrt{z}$. So

$$M = \iiint \delta dV = \int_0^{2\pi} \int_2^4 \int_0^{\sqrt{z}} z r dr dz d\theta = 2\pi \int_2^4 z \left[\frac{r^2}{2} \right]_0^{\sqrt{z}} dz = \pi \int_2^4 z^2 dz = \pi \frac{z^3}{3} \Big|_2^4 = \pi \frac{64 - 8}{3} = \frac{56\pi}{3}$$

8. Find the center of mass of the piece of the solid paraboloid $z = x^2 + y^2$ for $2 \leq z \leq 4$ if the density is $\delta = z$.

- a. $\frac{14}{75}$
- b. $\frac{14}{15}$
- c. $\frac{45}{14}$ Correct Choice
- d. $\frac{14}{45}$
- e. $\frac{15}{14}$

Solution: By symmetry, $\bar{x} = \bar{y} = 0$. To find \bar{z} we compute:

$$M_z = \iiint z \delta dV = \int_0^{2\pi} \int_2^4 \int_0^{\sqrt{z}} z^2 r dr dz d\theta = 2\pi \int_2^4 z^2 \left[\frac{r^2}{2} \right]_0^{\sqrt{z}} dz = \pi \int_2^4 z^3 dz = \pi \frac{z^4}{4} \Big|_2^4 = \pi \frac{256 - 16}{4} = 60\pi$$

$$\bar{z} = \frac{M_z}{M} = 60\pi \frac{3}{56\pi} = \frac{45}{14}$$

9. Find the equation of the plane tangent to the hyperboloid $xyz = 6$ at the point $(3, 2, 1)$.

- a. $(x, y, z) = (3 + 2t, 2 + 3t, 1 + 6t)$
- b. $(x, y, z) = (2 + 3t, 3 + 2t, 6 + t)$
- c. $3x + 2y + z = 14$
- d. $3x + 2y + z = 18$
- e. $2x + 3y + 6z = 18$ Correct Choice

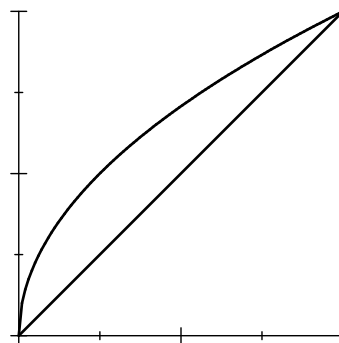
Solution: $P = (3, 2, 1)$ $F = xyz$ $\vec{\nabla}F = (yz, xz, xy)$ $\vec{N} = \vec{\nabla}F|_{(3,2,1)} = (2, 3, 6)$

$$N \cdot X = N \cdot P \quad 2x + 3y + 6z = 2(3) + 3(2) + 6(1) = 18$$

10. Find the volume under the surface $z = 2xy$ above the region bounded by $y = x$ and $y = 2\sqrt{x}$.

The base is shown at the right.

- a. $\frac{128}{3}$
- b. $\frac{128}{5}$
- c. $\frac{64}{3}$ Correct Choice
- d. $\frac{64}{5}$
- e. $\frac{64}{7}$

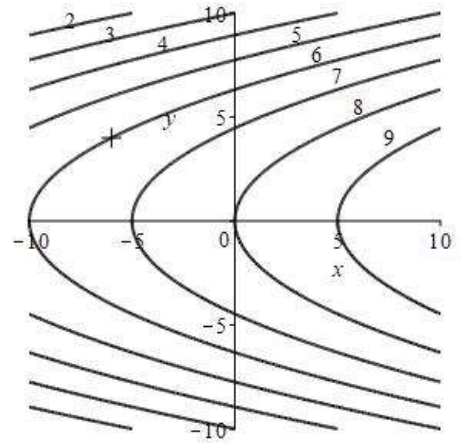


Solution: The curves intersect when $x = 2\sqrt{x}$ or $x^2 = 4x$ or $x = 0, 4$

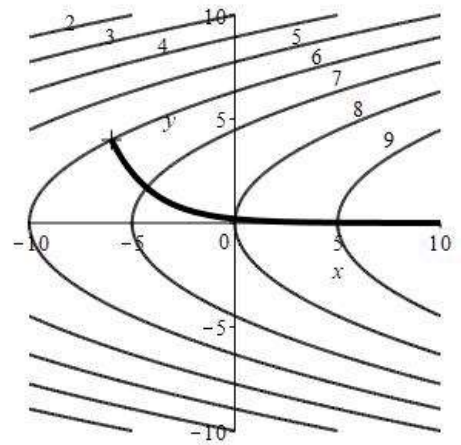
$$V = \int_0^4 \int_x^{2\sqrt{x}} 2xy dy dx = \int_0^4 [xy^2]_{y=x}^{2\sqrt{x}} dx = \int_0^4 (4x^2 - x^3) dx = \left[\frac{4x^3}{3} - \frac{x^4}{4} \right]_{x=0}^4 = 4^4 \left(\frac{4-3}{12} \right) = \frac{64}{3}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (5 points) At the right is the contour plot of a function $f(x,y)$. The contours are labeled by the function values. If you start at the cross at $(-6,4)$ and move so that your velocity is always in the direction of $\vec{\nabla}f$, the gradient of f , roughly sketch your path on the plot.

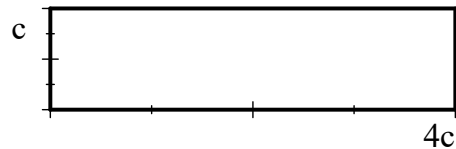
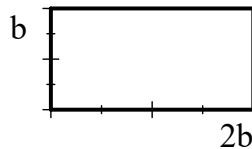
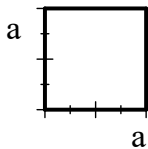


Solution: You are to draw a curve which starts at the cross, comes down and curves to the right, always perpendicular to each contour it crosses.



12. (20 points) A 118 cm wire is cut into 3 pieces. As shown in the plots, one piece is bent into a square of side a . Another piece is bent into a rectangle with sides b and $2b$. The third piece is bent into a rectangle with sides c and $4c$. Note: a , b and c may not be the same length, even if they look that way in the plots. Find a , b and c which minimize the total area enclosed in the three shapes. Note: the constraint is the sum of the perimeters.

You do NOT need to check it is a minimum rather than a maximum.



Solution: Minimize $A = a^2 + 2b^2 + 4c^2$ subject to the constraint $L = 4a + 6b + 10c = 118$.

Lagrange Multiplier: $\vec{\nabla}A = \langle 2a, 4b, 8c \rangle$ $\vec{\nabla}L = \langle 4, 6, 10 \rangle$ Lagrange equations: $\vec{\nabla}A = \lambda \vec{\nabla}L$

$$2a = 4\lambda \quad 4b = 6\lambda \quad 8c = 10\lambda \quad \text{or} \quad a = 2\lambda \quad b = \frac{3}{2}\lambda \quad c = \frac{5}{4}\lambda$$

Plug into the constraint:

$$4a + 6b + 10c = 8\lambda + 9\lambda + \frac{25}{2}\lambda = 118$$

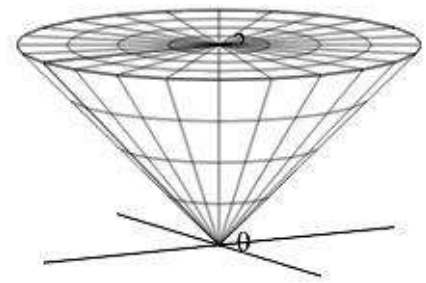
Solve for λ and substitute back:

$$16\lambda + 18\lambda + 25\lambda = 236 \quad \lambda = \frac{236}{59} = 4 \quad a = 8 \quad b = 6 \quad c = 5$$

13. (25 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$

for the vector field $\vec{F} = \langle -xz^2, -yz^2, z^3 \rangle$ and the solid above the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 2$.

Be careful with orientations. Use the following steps:



First the Left Hand Side:

a. Compute the divergence and give the volume element in the appropriate coordinate system:

$$\vec{\nabla} \cdot \vec{F} = -z^2 - z^2 + 3z^2 = z^2 \qquad dV = r \, dr \, d\theta \, dz$$

b. Compute the left hand side: Here are 2 ways:

$$\iiint_V \vec{\nabla} \cdot \vec{F} \, dV = \int_0^{2\pi} \int_0^2 \int_0^z z^2 r \, dr \, dz \, d\theta = 2\pi \int_0^2 \left[z^2 \frac{r^2}{2} \right]_{r=0}^z dz = \pi \int_0^2 z^4 \, dz = \pi \left[\frac{z^5}{5} \right]_0^2 = \frac{32\pi}{5}$$

$$\begin{aligned} \iiint_V \vec{\nabla} \cdot \vec{F} \, dV &= \int_0^{2\pi} \int_0^2 \int_r^2 z^2 r \, dz \, dr \, d\theta = 2\pi \int_0^2 \left[\frac{z^3}{3} r \right]_{z=r}^2 dr = \frac{2\pi}{3} \int_0^2 (8r - r^4) \, dr \\ &= \frac{2\pi}{3} \left[4r^2 - \frac{r^5}{5} \right]_0^2 = \frac{2\pi}{3} 2^4 \left(1 - \frac{2}{5} \right) = \frac{32\pi}{5} \end{aligned}$$

Second the Right Hand Side:

The boundary surface consists of the cone C and a disk D with appropriate orientations.

c. Parametrize the disk D :

$$\vec{R}(r, \theta) = \langle r \cos \theta, r \sin \theta, 2 \rangle$$

d. Compute the tangent vectors:

$$\vec{e}_r = \langle \cos \theta, \sin \theta, 0 \rangle$$

$$\vec{e}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

e. Compute the normal vector:

$$\vec{N} = \hat{i}(0) - \hat{j}(0) + \hat{k}(r \cos^2 \theta - -r \sin^2 \theta) = \langle 0, 0, r \rangle \quad \text{Need up. The orientation is correct.}$$

f. Evaluate $\vec{F} = \langle -xz^2, -yz^2, z^3 \rangle$ on the disk:

$$\vec{F} \Big|_{\vec{R}(r, \theta)} = \langle -4r \cos \theta, -4r \sin \theta, 8 \rangle$$

g. Compute the dot product:

$$\vec{F} \cdot \vec{N} = 8r$$

h. Compute the flux through D :

$$\iint_D \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 8r \, dr \, d\theta = 2\pi [4r^2]_0^2 = 32\pi$$

Parametrize the cone C as $\vec{R}(r, \theta) = \langle r \cos \theta, r \sin \theta, r \rangle$

i. Compute the tangent vectors:

$$\vec{e}_r = \langle \cos \theta, \sin \theta, 1 \rangle$$

$$\vec{e}_\theta = \langle -r \sin \theta, r \cos \theta, 0 \rangle$$

j. Compute the normal vector:

$$\vec{N} = \hat{i}(-r \cos \theta) - \hat{j}(-r \sin \theta) + \hat{k}(r \cos^2 \theta - r \sin^2 \theta) = \langle -r \cos \theta, -r \sin \theta, r \rangle$$

This is in and up. We need out and down.

Reverse: $\vec{N} = \langle r \cos \theta, r \sin \theta, -r \rangle$

k. Evaluate $\vec{F} = \langle -xz^2, -yz^2, z^3 \rangle$ on the cone:

$$\vec{F}|_{\vec{R}(\theta, \phi)} = \langle -r^3 \cos \theta, -r^3 \sin \theta, r^3 \rangle$$

l. Compute the dot product:

$$\vec{F} \cdot \vec{N} = -r^4 \cos^2 \theta - r^4 \sin^2 \theta - r^4 = -2r^4$$

m. Compute the flux through C :

$$\iint_C \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 -2r^4 dr d\theta = 2\pi \left[\frac{-2r^5}{5} \right]_0^2 = \frac{-128\pi}{5}$$

n. Compute the **TOTAL** right hand side:

$$\iint_{\partial V} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} + \iint_C \vec{F} \cdot d\vec{S} = 32\pi - \frac{128\pi}{5} = \frac{32\pi}{5} \quad \text{which agrees with (c).}$$