Name________________________

1-9	/45	12	/15
10	/5	13	/28
11	/10	Total	/103

MATH 251 Exam 1 Version A Fall 2020

Sections 517 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. The points $A = (1,2,3)$ and $B = (25,10,9)$ are the endpoints of the diameter of a sphere. If $C = (a,b,c)$ is the center and r is the radius, what is $a+b+c+r$?

a. 38 Correct Choice

b. 51

c. 64

d. 76

e. 194

Solution: The center is $C = \dfrac{A+B}{2} = (13,6,6)$ and the radius is
$r = D(A,C) = \sqrt{(13-1)^2 + (6-2)^2 + (6-3)^2} = \sqrt{144+16+9} = 13.$
So $a+b+c+r = 13+6+6+13 = 38.$

2. If $\vec{u}$ points Up and $\vec{v}$ points NorthEast, in what direction does $\vec{u}\times\vec{v}$ point?

a. SouthEast

b. SouthWest

c. NorthWest Correct Choice

d. Down

Solution: Aim the pointer finger of your right hand Up ($\vec{u}$) with your middle finger NorthEast ($\vec{v}$). Then your thumb points NorthWest ($\vec{u}\times\vec{v}$).

3. The Galactic Federation is trying to keep a stasis pod stationary in intergalactic space where there is no gravity. They already have 2 tractor beams pulling on the pod with the forces

$$\vec{F}_1 = \langle 3,1,2\rangle \qquad \vec{F}_2 = \langle -2,4,1\rangle$$

They now apply a 3^{rd} tractor beam with the force, $\vec{F}_3 = \langle a,b,c\rangle$, to keep the pod stationary. What is $a+b+c$?

a. -9 Correct Choice

b. -1

c. 0

d. 1

e. 9

Solution: $\vec{F}_3 = -\vec{F}_1 - \vec{F}_2 = -\langle 3,1,2\rangle - \langle -2,4,1\rangle = \langle -1,-5,-3\rangle$ So $a+b+c = -9$

4. The thrusters on the Starship Galileo exert the force $\vec{F} = \langle 2,3,-1 \rangle$ which moves the ship from $P = (4,3,5)$ to $Q = (5,4,3)$. Find the work done by the thrusters.

a. $W = 1$

b. $W = 3$

c. $W = 5$

d. $W = 7$ Correct Choice

e. $W = 9$

Solution: The displacement is $\overrightarrow{PQ} = Q - P = (1,1,-2)$. So the work done is $W = \vec{F} \cdot \overrightarrow{PQ} = 2 + 3 + 2 = 7$

5. Find the tangent vector, $\vec{v}$, to the curve $\vec{r}(t) = (t^3, t^2, t)$ at the point $(8,4,2)$. Then find its dot product with $\vec{F} = \langle 1,2,3 \rangle$.

a. $\vec{F} \cdot \vec{v} = (12,4,1)$

b. $\vec{F} \cdot \vec{v} = (12,8,3)$

c. $\vec{F} \cdot \vec{v} = 7$

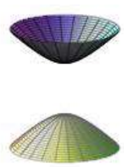
d. $\vec{F} \cdot \vec{v} = 17$

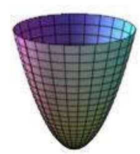
e. $\vec{F} \cdot \vec{v} = 23$ Correct Choice

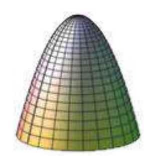
Solution: $\vec{v}(t) = \langle 3t^2, 2t, 1 \rangle$ $\vec{v}(2) = \langle 12,4,1 \rangle$ $\vec{F} \cdot \vec{v} = (1)(12) + (2)(4) + (3)(1) = 23$

6. Which of the following is the graph of the equation $z^2 = 4 + (x-1)^2 + (y-3)^2$?

a 

b 

c 

d 

Correct Choice (b)

Solution: $z^2 - (x-1)^2 - (y-3)^2 = 4$ is a hyperboloid of 2 sheets with axis parallel to the z-axis.

7. A point has spherical coordinates $(\rho,\phi,\theta) = \left(\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{6}\right)$. If its rectangular coordinates are (x,y,z), then $xyz =$

a. $\frac{3}{4}$

b. $\frac{3}{2}$

c. $\frac{\sqrt{3}}{2}$

d. $\frac{\sqrt{3}}{4}$ Correct Choice

e. $\frac{\sqrt{3}}{8}$

Solution:
$$x = \rho\sin\phi\cos\theta = \sqrt{2}\sin\frac{\pi}{4}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad xyz = \frac{\sqrt{3}}{2}\frac{1}{2}1 = \frac{\sqrt{3}}{4}$$
$$y = \rho\sin\phi\sin\theta = \sqrt{2}\sin\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{1}{2}$$
$$z = \rho\cos\phi = \sqrt{2}\cos\frac{\pi}{4} = 1$$

8. Find the area of the triangle with two edges $\vec{v} = \langle -2,1,3\rangle$ and $\vec{w} = \langle 1,0,2\rangle$.

a. $A = \frac{1}{2}\sqrt{27}$

b. $A = \sqrt{27}$

c. $A = \frac{1}{2}\sqrt{54}$ Correct Choice

d. $A = \sqrt{54}$

e. $A = 27$

Solution:
$$\vec{v}\times\vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2 & 1 & 3 \\ 1 & 0 & 2 \end{vmatrix} = \hat{\imath}(2-0) - \hat{\jmath}(-4-3) + \hat{k}(0-1) = \langle 2,7,-1\rangle$$
$$A = \frac{1}{2}|\vec{v}\times\vec{w}| = \frac{1}{2}\sqrt{4+49+1} = \frac{1}{2}\sqrt{54}$$

9. Find the volume of the parallelepiped with edges $\vec{u} = \langle 3,-2,1\rangle$, $\vec{v} = \langle -2,1,3\rangle$ and $\vec{w} = \langle 1,0,2\rangle$.

a. $V = 19$

b. $V = 9$ Correct Choice

c. $V = \frac{9}{2}$

d. $V = -9$

e. $V = -19$

Solution: $\vec{u}\cdot\vec{v}\times\vec{w} = \langle 3,-2,1\rangle\cdot\langle 2,7,-1\rangle = 6 - 14 - 1 = -9 \qquad V = |\vec{u}\cdot\vec{v}\times\vec{w}| = 9$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (5 points) Find a parametric equation of the line which is perpendicular to the plane $3x + 2y - z = 4$ and passes through the point $(3, 5, 1)$.

Solution: The direction of the line is the normal to the plane:

$$\vec{v} = \vec{N} = \langle 3, 2, -1 \rangle$$

A point on the line is $P = (3, 5, 1)$. So the line is

$$X = P + tv = (3, 5, 1) + t\langle 3, 2, -1 \rangle$$

or

$$x = 3 + 3t \qquad y = 5 + 2t \qquad z = 1 - t$$

11. (10 points) Find a normal equation of the plane which contains the line $(x, y, z) = (3 - 2t, 2 + t, 2 + 2t)$ and passes through the point $(3, 4, 1)$.

Solution: Two points on the plane are the point on the line and the given point:

$$P = (3, 2, 2) \quad \text{and} \quad Q = (3, 4, 1)$$

So one tangent vector is:

$$\vec{u} = \overrightarrow{PQ} = Q - P = \langle 0, 2, -1 \rangle$$

Another tangent vector is the tangent to the line:

$$\vec{v} = \langle -2, 1, 2 \rangle$$

The normal to the plane is the cross product of the two tangent vectors:

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & -1 \\ -2 & 1 & 2 \end{vmatrix} = \hat{\imath}(4 + 1) - \hat{\jmath}(0 - 2) + \hat{k}(0 + 4) = \langle 5, 2, 4 \rangle$$

So an equation of the plane is $\vec{N} \cdot X = \vec{N} \cdot P$, or:

$$5x + 2y + 4z = 5(3) + 2(2) + 4(2) = 27$$

Any multiple of this is OK.

12. (15 points) Consider the two planes

$$y + z = 3$$
$$x + 2y + z = 4$$

a. (4 pts) Find the angle (in degrees) between the planes.

Solution: The normal vectors are: $\vec{N}_1 = \langle 0,1,1 \rangle$ and $\vec{N}_2 = \langle 1,2,1 \rangle$. The cosine of the angle between them is:

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{|\vec{N}_1|\,|\vec{N}_2|} = \frac{2+1}{\sqrt{1+1}\,\sqrt{1+4+1}} = \frac{3}{\sqrt{2}\,\sqrt{6}} = \frac{\sqrt{3}}{2}$$

So $\theta = 30°$

b. (4 pts) Find a direction vector, $\vec{v}$, for the line of intersection of the planes.

Solution: The direction of the line is perpendicular to both normals. So

$$\vec{v} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \hat{\imath}(1-2) - \hat{\jmath}(0-1) + \hat{k}(0-1) = \langle -1,1,-1 \rangle$$

Any multiple of this is acceptable.

c. (4 pts) Find a point, P, on the line of intersection of the planes.

Solution: If we look for the solution with $z = 0$, then the equations reduce to $y = 3$ and $x + 2y = 4$. So $x = -2$ and a point is $P = (-2,3,0)$.
If we look for the solution with $x = 0$, then the equations reduce to $y + z = 3$ and $2y + z = 4$. We subtract the equations to see $y = 1$. Then $z = 2$ and a point is $P = (0,1,2)$.
If we look for the solution with $y = 0$, then the equations reduce to $z = 3$ and $x + z = 4$. So $x = 1$ and a point is $P = (1,0,3)$.
There are many other correct answers.

d. (3 pts) Find a parametric equation for the line of intersection of the planes.

Solution: There are many correct answers, depending on the P and $\vec{v}$ you found. Here is one:

$$X = P + t\vec{v} = (-2,3,0) + t\langle -1,1,-1 \rangle$$

or

$$x = -2 - t \qquad y = 3 + t \qquad z = -t$$

13. (28 points) For the parametric curve $\vec{r}(t) = \left(\frac{1}{3}t^3, t^2, 2t\right)$ compute each of the following:

a. (3 pts) velocity $\vec{v}$

Solution: Differentiate $\vec{r}$: $\vec{v} = \underline{(t^2, 2t, 2)}$

b. (3 pts) acceleration $\vec{a}$

Solution: Differentiate $\vec{v}$: $\vec{a} = \underline{(2t, 2, 0)}$

c. (3 pts) jerk $\vec{j}$

Solution: Differentiate $\vec{a}$: $\vec{j} = \underline{(2, 0, 0)}$

d. (3 pts) speed $|\vec{v}|$ (Simplify!)
HINT: The quantity inside the square root is a perfect square.

Solution: $|\vec{v}| = \sqrt{t^4 + 4t^2 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$ $|\vec{v}| = \underline{t^2 + 2}$

e. (2 pts) tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(t^2 + 2) = 2t$ $a_T = \underline{2t}$

f. (2 pts) the values of t where the curve passes thru the points

$A = \left(\frac{1}{3}, 1, 2\right)$ $t = \underline{1}$

$B = (9, 9, 6)$ $t = \underline{3}$

Solution: Compare each point to the curve $\left(\frac{1}{3}t^3, t^2, 2t\right)$. The y component is sufficient, but you should check the other components.

g. (4 pts) arc length between $\left(\frac{1}{3}, 1, 2\right)$ and $(9, 9, 6)$

Solution: $L = \int_{(1/3,1,2)}^{(9,9,6)} ds = \int_1^3 |\vec{v}|\,dt = \int_1^3 (t^2 + 2)\,dt = \left[\frac{t^3}{3} + 2t\right]_1^3$
$= (15) - \left(\frac{1}{3} + 2\right) = \frac{38}{3}$ $L = \underline{\frac{38}{3}}$

h. (4 pts) A wire has the shape of this curve between $\left(\frac{1}{3}, 1, 2\right)$ and $(9, 9, 6)$. Find the mass of the wire if the linear mass density is $\delta = 3yz$.

Solution: $|\vec{v}| = t^2 + 2$ $\delta = 3yz = 3t^2 2t = 6t^3$
$M = \int_{(1/3,1,2)}^{(9,9,6)} \delta\,ds = \int_1^3 3yz|\vec{v}|\,dt = \int_1^3 6t^3(t^2 + 2)\,dt = \int_1^3 (6t^5 + 12t^3)\,dt = \left[t^6 + 3t^4\right]_1^3$
$= (3^6 + 3^5) - (1 + 3) = 968$ $M = \underline{968}$

i. (4 pts) A wire has the shape of this curve. Find the work done by the force $\vec{F} = (0, z, -y)$ which pushes a bead along the wire from $\left(\frac{1}{3}, 1, 2\right)$ to $(9, 9, 6)$.

Solution: $\vec{F} = (0, z, -y) = (0, 2t, -t^2)$ $\vec{v} = (t^2, 2t, 2)$ $\vec{F} \cdot \vec{v} = 4t^2 - 2t^2 = 2t^2$
$W = \int_{(1/3,1,2)}^{(9,9,6)} \vec{F} \cdot d\vec{s} = \int_1^3 \vec{F} \cdot \vec{v}\,dt = \int_1^3 2t^2\,dt = \left[\frac{2t^3}{3}\right]_1^3 = 18 - \frac{2}{3} = \frac{52}{3}$ $W = \underline{\frac{52}{3}}$