Name________________________

1-9	/45	12	/15
10	/5	13	/28
11	/10	Total	/103

MATH 251 Exam 1 Version B Fall 2020

Sections 519 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. The points $A=(3,2,1)$ and $B=(5,6,5)$ are the endpoints of the diameter of a sphere. If $C=(a,b,c)$ is the center and r is the radius, what is $a+b+c+r$?

a. 40

b. 31

c. 29

d. 20

e. 14 Correct Choice

Solution: The center is $C=\dfrac{A+B}{2}=(4,4,3)$ and the radius is
$r=D(A,C)=\sqrt{(4-3)^2+(4-2)^2+(3-1)^2}=\sqrt{1+4+4}=3.$
So $a+b+c+r=4+4+3+3=14$.

2. If $\vec{u}$ points NorthWest and $\vec{v}$ points Down, in what direction does $\vec{u}\times\vec{v}$ point?

a. SouthEast

b. SouthWest Correct Choice

c. NorthWest

d. Up

Solution: Aim the pointer finger of your right hand NorthWest ($\vec{u}$) with your middle finger Down ($\vec{v}$). Then your thumb points SouthWest ($\vec{u}\times\vec{v}$).

3. The Galactic Federation is trying to keep a stasis pod stationary in intergalactic space where there is no gravity. They already have 2 tractor beams pulling on the pod with the forces

$$\vec{F}_1=\langle -3,2,1\rangle \qquad \vec{F}_2=\langle -1,-2,3\rangle$$

They now apply a 3^{rd} tractor beam with the force, $\vec{F}_3=\langle a,b,c\rangle$, to keep the pod stationary. What is $a+b+c$?

a. -8

b. -2

c. 0 Correct Choice

d. 2

e. 8

Solution: $\vec{F}_3=-\vec{F}_1-\vec{F}_2=-\langle -3,2,1\rangle-\langle -1,-2,3\rangle=\langle 4,0,-4\rangle$ So $a+b+c=0$

4. The thrusters on the Starship Galileo exert the force $\vec{F} = \langle 4,-2,1\rangle$ which moves the ship from $P = (1,5,2)$ to $Q = (3,4,5)$. Find the work done by the thrusters.

a. $W = 15$

b. $W = 13$ Correct Choice

c. $W = 11$

d. $W = 9$

e. $W = 7$

Solution: The displacement is $\overrightarrow{PQ} = Q - P = (2,-1,3)$. So the work done is $W = \vec{F}\bullet\overrightarrow{PQ} = 8 + 2 + 3 = 13$

5. Find the tangent vector, $\vec{v}$, to the curve $\vec{r}(t) = (t^3,t^2,t)$ at the point $(8,4,2)$. Then find it dot product with $\vec{F} = \langle 3,2,1\rangle$.

a. $\vec{F}\bullet\vec{v} = (12,4,1)$

b. $\vec{F}\bullet\vec{v} = (36,8,1)$

c. $\vec{F}\bullet\vec{v} = 19$

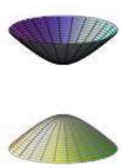
d. $\vec{F}\bullet\vec{v} = 29$

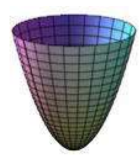
e. $\vec{F}\bullet\vec{v} = 45$ Correct Choice

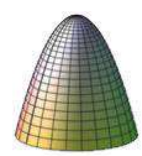
Solution: $\vec{v}(t) = \langle 3t^2,2t,1\rangle$ $\vec{v}(2) = \langle 12,4,1\rangle$ $\vec{F}\bullet\vec{v} = (3)(12) + (2)(4) + (1)(1) = 45$

6. Which of the following is the graph of the equation $(x-1)^2 + (y-3)^2 - z = 4$?

a 

b 

c 

d 

Correct Choice (c)

Solution: $z = -4 + (x-1)^2 + (y-3)^2$ is an elliptic paraboloid opening up in the z-direction.

7. A point has spherical coordinates $(\rho, \phi, \theta) = \left(\sqrt{2}, \frac{\pi}{6}, \frac{\pi}{4}\right)$. If its rectangular coordinates are (x, y, z), then $xyz =$

a. $\frac{\sqrt{6}}{8}$ Correct Choice

b. $\frac{\sqrt{6}}{4}$

c. $\frac{\sqrt{3}}{4}$

d. $\frac{3\sqrt{2}}{8}$

e. $\frac{3\sqrt{2}}{4}$

Solution:
$$x = \rho \sin\phi \cos\theta = \sqrt{2} \sin\frac{\pi}{6} \cos\frac{\pi}{4} = \frac{1}{2} \qquad xyz = \frac{1}{2}\frac{1}{2}\frac{\sqrt{6}}{2} = \frac{\sqrt{6}}{8}$$
$$y = \rho \sin\phi \sin\theta = \sqrt{2} \sin\frac{\pi}{6} \sin\frac{\pi}{4} = \frac{1}{2}$$
$$z = \rho \cos\phi = \sqrt{2} \cos\frac{\pi}{6} = \frac{\sqrt{6}}{2}$$

8. Find the area of the triangle with two edges $\vec{v} = \langle 4, 0, 1 \rangle$ and $\vec{w} = \langle 2, 1, -1 \rangle$.

a. $A = \frac{1}{2}\sqrt{53}$ Correct Choice

b. $A = \sqrt{53}$

c. $A = \frac{1}{2}\sqrt{27}$

d. $A = \sqrt{27}$

e. $A = 27$

Solution: $$\vec{v} \times \vec{w} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 4 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \hat{\imath}(0 - 1) - \hat{\jmath}(-4 - 2) + \hat{k}(4 - 0) = \langle -1, 6, 4 \rangle$$
$$A = \frac{1}{2}|\vec{v} \times \vec{w}| = \frac{1}{2}\sqrt{1 + 36 + 16} = \frac{1}{2}\sqrt{53}$$

9. Find the volume of the parallelepiped with edges $\vec{u} = \langle 2, -3, 1 \rangle$, $\vec{v} = \langle 4, 0, 1 \rangle$ and $\vec{w} = \langle 2, 1, -1 \rangle$.

a. $V = -20$

b. $V = -16$

c. $V = 8$

d. $V = 16$ Correct Choice

e. $V = 20$

Solution: $\vec{u} \cdot \vec{v} \times \vec{w} = \langle 2, -3, 1 \rangle \cdot \langle -1, 6, 4 \rangle = -2 - 18 + 4 = -16$ $\qquad V = |\vec{u} \cdot \vec{v} \times \vec{w}| = 16$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (5 points) Find a parametric equation of the line which is perpendicular to the plane $3x + 2y - z = 4$ and passes through the point $(3,5,1)$.

Solution: The direction of the line is the normal to the plane:

$$\vec{v} = \vec{N} = \langle 3,2,-1 \rangle$$

A point on the line is $P = (3,5,1)$. So the line is

$$X = P + tv = (3,5,1) + t\langle 3,2,-1 \rangle$$

or

$$x = 3 + 3t \qquad y = 5 + 2t \qquad z = 1 - t$$

11. (10 points) Find a normal equation of the plane which contains the line $(x,y,z) = (3-2t, 2+t, 2+2t)$ and passes through the point $(3,4,1)$.

Solution: Two points on the plane are the point on the line and the given point:

$$P = (3,2,2) \quad \text{and} \quad Q = (3,4,1)$$

So one tangent vector is:

$$\vec{u} = \overrightarrow{PQ} = Q - P = \langle 0,2,-1 \rangle$$

Another tangent vector is the tangent to the line:

$$\vec{v} = \langle -2,1,2 \rangle$$

The normal to the plane is the cross product of the two tangent vectors:

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & -1 \\ -2 & 1 & 2 \end{vmatrix} = \hat{\imath}(4+1) - \hat{\jmath}(0-2) + \hat{k}(0+4) = \langle 5,2,4 \rangle$$

So an equation of the plane is $\vec{N} \cdot X = \vec{N} \cdot P$, or:

$$5x + 2y + 4z = 5(3) + 2(2) + 4(2) = 27$$

Any multiple of this is OK.

12. (15 points) Consider the two planes

$$y + z = 3$$
$$2x + 2y + z = 4$$

a. (4 pts) Find the angle (in degrees) between the planes.

Solution: The normal vectors are: $\vec{N}_1 = \langle 0,1,1 \rangle$ and $\vec{N}_2 = \langle 2,2,1 \rangle$. The cosine of the angle between them is:

$$\cos\theta = \frac{\vec{N}_1 \cdot \vec{N}_2}{\left|\vec{N}_1\right|\left|\vec{N}_2\right|} = \frac{2+1}{\sqrt{1+1}\,\sqrt{4+4+1}} = \frac{3}{\sqrt{2}\,3} = \frac{1}{\sqrt{2}}$$

So $\theta = 45°$

b. (4 pts) Find a direction vector, $\vec{v}$, for the line of intersection of the planes.

Solution: The direction of the line is perpendicular to both normals. So

$$\vec{v} = \vec{N}_1 \times \vec{N}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 2 & 2 & 1 \end{vmatrix} = \hat{\imath}(1-2) - \hat{\jmath}(0-2) + \hat{k}(0-2) = \langle -1,2,-2 \rangle$$

Any multiple of this is acceptable.

c. (4 pts) Find a point, P, on the line of intersection of the planes.

Solution: If we look for the solution with $z = 0$, then the equations reduce to $y = 3$ and $2x + 2y = 4$. So $x = -1$ and a point is $P = (-1,3,0)$.
If we look for the solution with $x = 0$, then the equations reduce to $y + z = 3$ and $2y + z = 4$. We subtract the equations to see $y = 1$. Then $z = 2$ and a point is $P = (0,1,2)$.
If we look for the solution with $y = 0$, then the equations reduce to $z = 3$ and $2x + z = 4$. So $x = \frac{1}{2}$ and a point is $P = \left(\frac{1}{2},0,3\right)$.
There are many other correct answers.

d. (3 pts) Find a parametric equation for the line of intersection of the planes.

Solution: There are many correct answers, depending on the P and $\vec{v}$ you found. Here is one:

$$X = P + t\vec{v} = (-1,3,0) + t\langle -1,2,-2 \rangle$$

or

$$x = -1 - t \qquad y = 3 + 2t \qquad z = -2t$$

13. (28 points) For the parametric curve $\vec{r}(t) = \left(t^2, \frac{1}{3}t^3, 2t\right)$ compute each of the following:

a. (3 pts) velocity $\vec{v}$

Solution: Differentiate $\vec{r}$: $\vec{v} =$ $(2t, t^2, 2)$

b. (3 pts) acceleration $\vec{a}$

Solution: Differentiate $\vec{v}$: $\vec{a} =$ $(2, 2t, 0)$

c. (3 pts) jerk $\vec{j}$

Solution: Differentiate $\vec{a}$: $\vec{j} =$ $(0, 2, 0)$

d. (2 pts) speed $|\vec{v}|$ (Simplify!)
HINT: The quantity inside the square root is a perfect square.

Solution: $|\vec{v}| = \sqrt{4t^2 + t^4 + 4} = \sqrt{(t^2+2)^2} = t^2 + 2$ $\qquad |\vec{v}| =$ $t^2 + 2$

e. (2 pts) tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}(t^2 + 2) = 2t$ $\qquad a_T =$ $2t$

f. (2 pts) the values of t where the curve passes thru the points

$A = \left(1, \frac{1}{3}, 2\right)$ $\qquad t =$ 1

$B = (9, 9, 6)$ $\qquad t =$ 3

Solution: Compare each point to the curve $\left(t^2, \frac{1}{3}t^3, 2t\right)$. The x component is sufficient, but you should check the other components.

g. (4 pts) arc length between $\left(1, \frac{1}{3}, 2\right)$ and $(9, 9, 6)$

Solution: $L = \int_{(1,1/3,2)}^{(9,9,6)} ds = \int_1^3 |\vec{v}|\,dt = \int_1^3 (t^2 + 2)dt = \left[\frac{t^3}{3} + 2t\right]_1^3$

$= (15) - \left(\frac{1}{3} + 2\right) = \frac{38}{3}$ $\qquad L =$ $\frac{38}{3}$

h. (4 pts) A wire has the shape of this curve between $\left(1, \frac{1}{3}, 2\right)$ and $(9, 9, 6)$. Find the mass of the wire if the linear mass density is $\delta = xz$.

Solution: $|\vec{v}| = t^2 + 2 \qquad \delta = xz = t^2 2t = 2t^3$

$M = \int_{(1,1/3,2)}^{(9,9,6)} \delta\,ds = \int_1^3 xz|\vec{v}|\,dt = \int_1^3 2t^3(t^2+2)\,dt = \int_1^3 (2t^5 + 4t^3)\,dt = \left[\frac{t^6}{3} + t^4\right]_1^3$

$= (3^5 + 3^4) - \left(\frac{1}{3} + 1\right) = \frac{968}{3}$ $\qquad M =$ $\frac{968}{3}$

i. (4 pts) A wire has the shape of this curve. Find the work done by the force $\vec{F} = (0, z, y)$ which pushes a bead along the wire from $\left(1, \frac{1}{3}, 2\right)$ to $(9, 9, 6)$.

Solution: $\vec{F} = (0, z, y) = \left(0, 2t, \frac{1}{3}t^3\right) \qquad \vec{v} = (2t, t^2, 2) \qquad \vec{F} \cdot \vec{v} = 2t^3 + \frac{2}{3}t^3 = \frac{8}{3}t^3$

$W = \int_{(1,1/3,2)}^{(9,9,6)} \vec{F} \cdot d\vec{s} = \int_1^3 \vec{F} \cdot \vec{v}\,dt = \int_1^3 \frac{8}{3}t^3\,dt = \left[\frac{2}{3}t^4\right]_1^3 = 54 - \frac{2}{3} = \frac{162 - 2}{3}$ $\qquad W =$ $\frac{160}{3}$