Name________________________

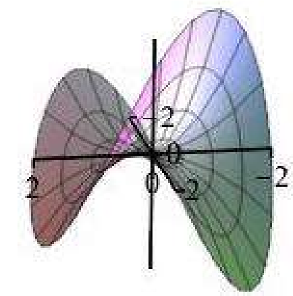
1-10	/50	12	/18
11	/12	13	/20+10EC
		Total	/100+10EC

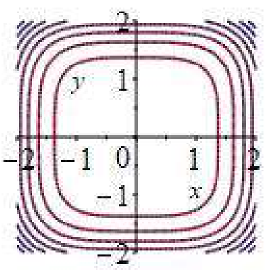
MATH 251 Exam 2 Version A Fall 2020

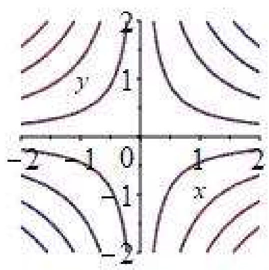
Sections 517 Solutions P. Yasskin

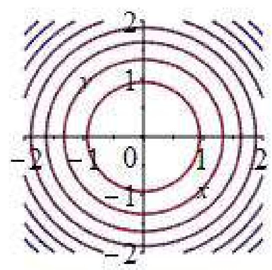
Multiple Choice: (5 points each. No part credit.)

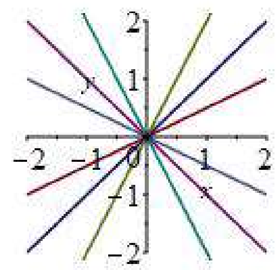
1. Which of the following is the contour plot for the function whose graph is shown at the right?



a


b


c


d


Correct Choice

Solution: The graph has 2 ups and 2 downs. So contour plot is (b).

2. Identify the domain and image of the function $w = f(x,y,z) = \ln(9 - x^2 - y^2 - z^2)$.
Be sure to indicate whether the boundary of any region is included or not.
Enter each answer as an inequality or an interval.

Solution: The domain is the set of points where the function is defined. This function is only defined if the quantity inside the $\ln$ is positive (and not 0). But it is also less than or equal to 9. So $0 < 9 - x^2 - y^2 - z^2 \le 9$ So the domain is the set of all points (x,y,z) such that:

$$\boxed{0 \le x^2 + y^2 + z^2 < 9}$$

The image is the set of all possible values of the function. The quantity inside the $\ln$ can be any number between 0 and 9, including 9 but not including 0. So the image is the set of all numbers w such that:

$$\boxed{-\infty < w \le \ln 9 \quad \text{or} \quad (-\infty, \ln 9]}$$

3. Find the tangent plane to the graph of $z = x^2y^3$ at the point $(x,y) = (2,1)$. Then find its z-intercept.

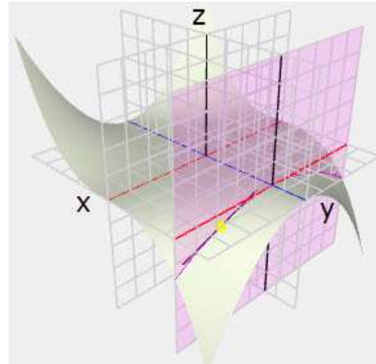
Solution:

$f = x^2y^3$ $\quad f(2,1) = 4$ $\quad z = f(2,1) + f_x(2,1)(x-1) + f_y(2,1)(y-1)$

$f_x = 2xy^3$ $\quad f_x(2,1) = 4$ $\quad \boxed{z = 4 + 4(x-2) + 12(y-1)}$

$f_y = 3x^2y^2$ $\quad f_y(2,1) = 12$ $\quad \boxed{z = 4x + 12y - 16}$ $\quad \boxed{z\text{-intercept} = -16}$

4. In the figure at the right, the vertical plane intersects the surface in a curve which is either the x-Trace or the y-Trace. Which is it? The slope of the tangent line to this trace is either the x-Partial Derivative or the y-Partial Derivative. Which is it?



Solution: Since y is constant on this curve, it is the $\boxed{x\text{-Trace}}$ and the slope of the tangent is the $\boxed{x\text{-Partial Derivative}}$.

5. Suppose $w = w(x,y,z)$ while $x = s^2$, $y = t^3$ and $z = s^3 + t^2$, find $\left.\frac{\partial w}{\partial t}\right|_{(1,1)}$ given that

$$\left.\frac{\partial w}{\partial x}\right|_{(1,1,2)} = 3 \qquad \left.\frac{\partial w}{\partial y}\right|_{(1,1,2)} = 4 \qquad \left.\frac{\partial w}{\partial z}\right|_{(1,1,2)} = 5$$

Solution:

$$\frac{\partial x}{\partial t} = 0 \qquad \frac{\partial y}{\partial t} = 3t^2 \qquad \frac{\partial z}{\partial t} = 2t$$

$$\left.\frac{\partial x}{\partial t}\right|_{(1,1)} = 0 \qquad \left.\frac{\partial y}{\partial t}\right|_{(1,1)} = 3 \qquad \left.\frac{\partial z}{\partial t}\right|_{(1,1)} = 2$$

$$\left.\frac{\partial w}{\partial t}\right|_{(1,1)} = \left.\frac{\partial w}{\partial x}\right|_{(1,1,2)}\left.\frac{\partial x}{\partial t}\right|_{(1,1)} + \left.\frac{\partial w}{\partial y}\right|_{(1,1,2)}\left.\frac{\partial y}{\partial t}\right|_{(1,1)} + \left.\frac{\partial w}{\partial z}\right|_{(1,1,2)}\left.\frac{\partial z}{\partial t}\right|_{(1,1)}$$

$$= 3 \cdot 0 + 4 \cdot 3 + 5 \cdot 2 = \boxed{22}$$

6. The equation $x^2z^3 + y^3z^2 = 17$ defines a surface which passes thru the point $(3,2,1)$. This surface implicitly defines a function $z = f(x,y)$ passing through this point. Find $\frac{\partial f}{\partial y}(3,2)$.

Solution: Apply $\frac{\partial}{\partial y}$ to both sides: $3x^2z^2\frac{\partial z}{\partial y} + 3y^2z^2 + 2y^3z\frac{\partial z}{\partial y} = 0.$

We plug in $(x,y,z) = (3,2,1)$: $27\frac{\partial z}{\partial y} + 12 + 16\frac{\partial z}{\partial y} = 0$ and solve $\frac{\partial z}{\partial y} = \boxed{-\frac{12}{43}}$

7. Find the equation of the tangent plane to the hyperboloid $3(x-3)^2 - (y-2)^2 - (z-4)^2 = 1$ at the point $(x,y,z) = (2,3,3)$.

Solution: Let $f = 3(x-3)^2 - (y-2)^2 - (z-4)^2$ and $P = (2,3,3)$.
Then $\vec{\nabla} f = \langle 6(x-3), -2(y-2), -2(z-4)\rangle$. $\vec{N} = \left.\vec{\nabla} f\right|_{(2,3,3)} = \langle -6,-2,2\rangle$

$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad -6x - 2y + 2z = -6(2) - 2(3) + 2(3) = -12$ $\quad \boxed{3x + y - z = 6}$

8. A cardboard box has length $L = 5\,cm$, width $W = 4\,cm$ and height $H = 3\,cm$. Use the linear approximation to estimate the volume of cardboard used to make this box if the thickness of cardboard on the bottom is $0.8\,cm$, the thickness of cardboard on the 4 sides is $0.4\,cm$, and the thickness of cardboard on the top is $0.2\,cm$.

Note: There are 2 thicknesses of cardboard in each direction.

Solution: The change in the height is the thickness of the top and bottom: $\Delta H = 0.8 + 0.2 = 1.0$
The change in the length and width is twice the thickness of the sides: $\Delta L = \Delta W = 2(0.4) = 0.8$

$$\Delta V \approx dV = \frac{\partial V}{\partial L}dL + \frac{\partial V}{\partial W}dW + \frac{\partial V}{\partial H}dH = WH\,dL + LH\,dW + LW\,dH$$
$$= (4)(3)(0.8) + (5)(3)(0.8) + (5)(4)(1.0) = 41.6 \qquad \boxed{\Delta V \approx 41.6}$$

9. If 2 resistors with resistances R_1 and R_2 are arranged in parallel, then the net resistance R is given by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Initially, $R_1 = 2\Omega$, $R_2 = 4\Omega$. First find the initial value of R. If R_1 is increasing at $\frac{dR_1}{dt} = 0.3\,\frac{\Omega}{\text{sec}}$ while R_2 is decreasing at $\frac{dR_2}{dt} = -0.3\,\frac{\Omega}{\text{sec}}$, find the rate at which the net resistance is changing $\frac{dR}{dt}$. Be careful to get the sign correct.

Enter exact numbers. For example, $\frac{-0.5}{7}$ should be entered as -0.5/7, NO SPACES, NO UNITS.

Solution: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \qquad R = \boxed{\frac{4}{3}}\,\Omega$

$$-\frac{1}{R^2}\frac{dR}{dt} = -\frac{1}{{R_1}^2}\frac{dR_1}{dt} - \frac{1}{{R_2}^2}\frac{dR_2}{dt}$$

$$\frac{dR}{dt} = \frac{R^2}{{R_1}^2}\frac{dR_1}{dt} + \frac{R^2}{{R_2}^2}\frac{dR_2}{dt} = \frac{16}{9}\frac{1}{4}(0.3) + \frac{16}{9}\frac{1}{16}(-0.3) = \frac{4}{9}0.3 - \frac{1}{9}0.3 = \frac{0.9}{9} \qquad \frac{dR}{dt} = \boxed{0.1}\,\frac{\Omega}{\text{sec}}$$

10. The point $(-4,1)$ is a critical point of the function $g(x,y) = x^2y - 2y^3x + 10xy$. Apply the 2^{nd}-Derivative Test to classify $(-4,1)$.

a. Local Minimum Correct Choice

b. Local Maximum

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $g_x = 2xy - 2y^3 + 10y \qquad g_y = x^2 - 6y^2x + 10x$

Check critical point: $g_x(-4,1) = -8 - 2 + 10 = 0 \qquad g_y = 16 + 24 - 40 = 0$

$g_{xx} = 2y \qquad g_{yy} = -12yx \qquad g_{xy} = 2x - 6y^2 + 10$

$g_{xx}(-4,1) = 2 \qquad g_{yy}(-4,1) = 48 \qquad g_{xy}(-4,1) = -4$

$D = g_{xx}g_{yy} - {g_{xy}}^2 = 96 - 16 = 80$

$D > 0$ and $g_{xx} > 0$ So this is a local minimum.

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (12 points) Find all 1^{st} and 2^{nd} partial derivatives of $f(x,y) = x^2 \sin(xy)$.
Enter f_{yy} here but put all of them on your paper.

Solution: $\boxed{f_x = 2x\sin(xy) + x^2 y\cos(xy)}$ $\boxed{f_y = x^3\cos(xy)}$

$\boxed{f_{xx} = 2\sin(xy) + 4xy\cos(xy) - x^2y^2\sin(xy)}$ $\boxed{f_{xy} = 3x^2\cos(xy) - x^3y\sin(xy)}$

$\boxed{f_{yx} = 3x^2\cos(xy) - x^3y\sin(xy)}$ $\boxed{f_{yy} = -x^4\sin(xy)}$

12. (18 points) The ideal gas law says the pressure, P, the density, δ, and the temperature, T, are related by $P = k\delta T$ where k is a constant. A weather balloon measures that at its current position,

$$P = .81\,atm \qquad \delta = 1.2\,\frac{kg}{m^3} \qquad T = 270°K$$

The weather balloon also measures that gradients of the density and temperature are

$$\vec{\nabla}\delta = \left\langle \frac{\partial\delta}{\partial x}, \frac{\partial\delta}{\partial y}, \frac{\partial\delta}{\partial z} \right\rangle = \langle .2, .1, -.2\rangle \frac{kg/m^3}{m}$$

$$\vec{\nabla}T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \langle 3, -12, 4\rangle \frac{°K}{m}$$

a. (2 pts) Find the constant k.

Solution: $k = \dfrac{P}{\delta T} = \dfrac{.81}{1.2 \bullet 270} = \dfrac{.01}{.4 \bullet 10} = \boxed{.0025}$

b. (9 pts) Find the gradient of the pressure.
HINT: Find each component separately. No need to simplify numbers.
Enter $\dfrac{\partial P}{\partial x}$ here but put all three components on your paper.

Solution:

$$\frac{\partial P}{\partial x} = kT\frac{\partial\delta}{\partial x} + k\delta\frac{\partial T}{\partial x} = .0025(270 \bullet .2 + 1.2 \bullet 3) = 0.144$$

$$\frac{\partial P}{\partial y} = kT\frac{\partial\delta}{\partial y} + k\delta\frac{\partial T}{\partial y} = .0025(270 \bullet .1 + 1.2 \bullet (-12)) = 0.0315$$

$$\frac{\partial P}{\partial z} = kT\frac{\partial\delta}{\partial z} + k\delta\frac{\partial T}{\partial z} = .0025(270 \bullet (-.2) + 1.2 \bullet 4) = -0.123$$

$$\boxed{\vec{\nabla}P = \langle 0.144, 0.0315, -0.123\rangle}$$

c. (3 pts) In the absence of wind or other forces, a balloon will tend to drift from regions of high density to regions of low density. In what unit vector direction, $\hat{u}$, will this balloon drift?

Solution: The direction from high density to low density is $\vec{u} = -\vec{\nabla}\delta = \langle -.2, -.1, .2\rangle$.

$$|\vec{u}| = \sqrt{(.2)^2 + (.1)^2 + (.2)^2} = \sqrt{.04 + .01 + .04} = \sqrt{.09} = .3$$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{.3}\langle -.2, -.1, .2\rangle = \boxed{\left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle}$$

d. (4 pts) If the balloon's current velocity is $\vec{v} = \langle 4, 1, 3\rangle \frac{m}{\sec}$, find the rate the temperature is changing as seen by the balloon.

Solution: $\dfrac{dT}{dt} = \vec{v} \bullet \vec{\nabla}T = \langle 4, 1, 3\rangle \bullet \langle 3, -12, 4\rangle = 12 - 12 + 12 = \boxed{12}\ \dfrac{°K}{\sec}$

13. (20 points + 10 points extra credit) A rectangular solid box has 3 faces in the coordinate planes and the remaining vertex on the plane $z = 36 - 3x - 4y$. Find the dimensions and volume of the largest such box.

NOTE: Solve by either Eliminating a Variable or by Lagrange Multipliers. Extra Credit for solving by both methods. Draw a line across your paper to clearly separate the two solutions.

Solution by Eliminating a Variable: We maximize the volume $V = xyz$ subject to the constraint $z = 36 - 3x - 4y$. So the volume becomes:

$$V = xy(36 - 3x - 4y) = 36xy - 3x^2y - 4xy^2$$

We find the partial derivatives,factor and set them equal to 0:

$$V_x = 36y - 6xy - 4y^2 = y(36 - 6x - 4y) = 0$$

$$V_y = 36x - 3x^2 - 8xy = x(36 - 3x - 8y) = 0$$

Since $x = 0$ or $y = 0$ give 0 volume, we can assume $x \neq 0$ and $y \neq 0$ and solve

$$6x + 4y = 36$$

$$3x + 8y = 36$$

Twice the first equation minus the second gives $9x = 36$ or $x = 4$.

Twice the second equation minus the first gives $12y = 36$ or $y = 3$.

Substituting back gives $z = 36 - 3x - 4y = 36 - 12 - 12 = 12$

So the dimensions are $\boxed{(x,y,z) = (4,3,12)}$ and the volume is $\boxed{V = 144}$.

Solution by Lagrange Multipliers: We maximize the volume $V = xyz$ subject to the constraint $g = 3x + 4y + z = 36$. The gradients are:

$$\nabla V = (yz, xz, xy) \qquad \nabla g = (3,4,1)$$

So the Lagrange equations are $\nabla C = \lambda \nabla V$ or

$$yz = \lambda 3 \qquad xz = \lambda 4 \qquad xy = \lambda$$

We plug $\lambda = xy$ into the other two equations:

$$yz = 3xy \quad \text{and} \quad xz = 4xy$$

These give $z = 3x$ and $z = 4y$. So the constraint becomes

$$36 = 3x + 4y + z = z + z + z = 3z \quad \text{or} \quad z = 12$$

Substiting back, we find $12 = 3x = 4y$. So $x = 4$ and $y = 3$.

So the dimensions are $\boxed{(x,y,z) = (4,3,12)}$ and the volume is $\boxed{V = 144}$.