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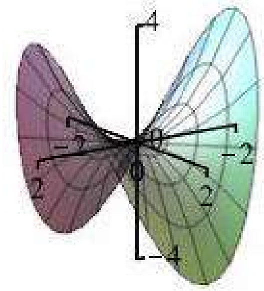
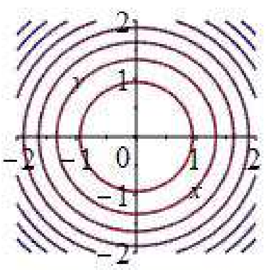
MATH 251 Exam 2 Version B Fall 2020
Sections 519 Solutions P. Yasskin

1-10	/50	12	/18
11	/12	13	/20+10EC
		Total	/100+10EC

Multiple Choice: (5 points each. No part credit.)

1. Which of the following is the contour plot for the function whose graph is shown at the right?



a


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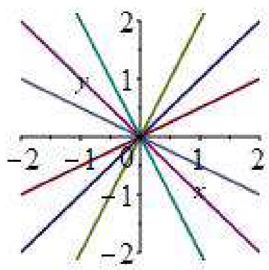
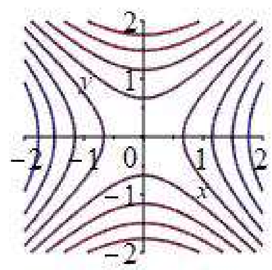
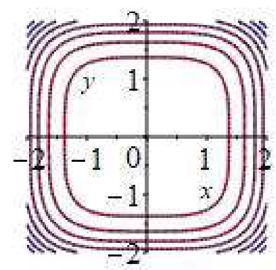


c


d


Correct Choice

Solution: The graph has 2 ups and 2 downs. So contour plot (c).

2. Identify the domain and image of the function $w = f(x,y,z) = \dfrac{2}{\sqrt{4 - x^2 - y^2 - z^2}}$.

Be sure to say whether the boundary of any region is included or not.
Enter each answer as an inequality or an interval.

Solution: The domain is the set of points where the function is defined. This function is not defined where the denominator is 0 or the quantity in the square root is negative. So the domain is the set of all points (x,y,z) such that:

$$\boxed{x^2 + y^2 + z^2 < 4}$$

The image is the set of all possible values of the function. The quantity inside the square root can be any number between 0 and 4, including 4 but not including 0 because then you would be dividing by 0. So the quotient can be any number between 1 and ∞. So the image is the set of all numbers w such that:

$$\boxed{1 \le w < \infty \quad \text{or} \quad [1,\infty)}$$

3. Find the tangent plane to the graph of $z = x^3y^2$ at the point $(x,y) = (1,2)$. Then find its z-intercept.

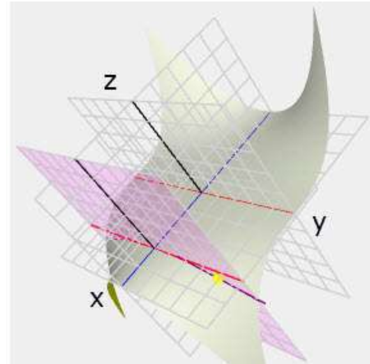
Solution:

$f = x^3y^2$ $\quad f(1,2) = 4$ $\quad z = f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2)$

$f_x = 3x^2y^2$ $\quad f_x(1,2) = 12$ $\quad \boxed{z = 4 + 12(x-1) + 4(y-2)}$

$f_y = 2x^3y$ $\quad f_y(1,2) = 4$ $\quad \boxed{z = 12x + 4y - 16}$ $\quad \boxed{z\text{-intercept} = -16}$

4. In the figure at the right, the vertical plane intersects the surface in a curve which is either the x-Trace or the y-Trace. Which is it? The slope of the tangent line to this trace is either the x-Partial Derivative or the y-Partial Derivative. Which is it?



Solution: Since x is constant on this curve, it is the $\boxed{y\text{-Trace}}$ and the slope of the tangent is the $\boxed{y\text{-Partial Derivative}}$.

5. Suppose $w = w(x,y,z)$ while $x = s^2$, $y = t^3$ and $z = s^3 + t^2$, find $\left.\frac{\partial w}{\partial s}\right|_{(1,1)}$ given that

$$\left.\frac{\partial w}{\partial x}\right|_{(1,1,2)} = 3 \qquad \left.\frac{\partial w}{\partial y}\right|_{(1,1,2)} = 4 \qquad \left.\frac{\partial w}{\partial z}\right|_{(1,1,2)} = 5$$

Solution:

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial y}{\partial s} = 0 \qquad \frac{\partial z}{\partial s} = 3s^2$$

$$\left.\frac{\partial x}{\partial s}\right|_{(1,1)} = 2 \qquad \left.\frac{\partial y}{\partial s}\right|_{(1,1)} = 0 \qquad \left.\frac{\partial z}{\partial s}\right|_{(1,1)} = 3$$

$$\left.\frac{\partial w}{\partial s}\right|_{(1,1)} = \left.\frac{\partial w}{\partial x}\right|_{(1,1,2)} \left.\frac{\partial x}{\partial s}\right|_{(1,1)} + \left.\frac{\partial w}{\partial y}\right|_{(1,1,2)} \left.\frac{\partial y}{\partial s}\right|_{(1,1)} + \left.\frac{\partial w}{\partial z}\right|_{(1,1,2)} \left.\frac{\partial z}{\partial s}\right|_{(1,1)}$$

$$= 3 \cdot 2 + 4 \cdot 0 + 5 \cdot 3 = \boxed{21}$$

6. The equation $x^2z^3 + y^3z^2 = 17$ defines a surface which passes thru the point $(3,2,1)$. This surface implicitly defines a function $z = f(x,y)$ passing through this point. Find $\frac{\partial f}{\partial x}(3,2)$.

Solution: Apply $\frac{\partial}{\partial x}$ to both sides: $2xz^3 + 3x^2z^2\frac{\partial z}{\partial x} + 2y^3z\frac{\partial z}{\partial x} = 0.$
We plug in $(x,y,z) = (3,2,1)$: $6 + 27\frac{\partial z}{\partial x} + 16\frac{\partial z}{\partial x} = 0$ and solve $\frac{\partial z}{\partial x} = \boxed{-\frac{6}{43}}$

7. Find the equation of the tangent plane to the hyperboloid $(x-3)^2 - (y-2)^2 + (z-4)^2 = 1$ at the point $(x,y,z) = (2,3,3)$.

Solution: Let $f = (x-3)^2 - (y-2)^2 + (z-4)^2$ and $P = (2,3,3)$.
Then $\vec{\nabla} f = \langle 2(x-3), -2(y-2), 2(z-4)\rangle$. $\vec{N} = \left.\vec{\nabla} f\right|_{(2,3,3)} = \langle -2,-2,-2\rangle$
$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad -2x - 2y - 2z = -2(2) - 2(3) - 2(3) = -16$ $\quad \boxed{x + y + z = 8}$

8. If 2 resistors with resistances R_1 and R_2 are arranged in parallel, then the net resistance R is given by:

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

Initially, $R_1 = 2\Omega$, $R_2 = 4\Omega$. First find the initial value of R. If R_1 increases by $\Delta R_1 = 0.3\Omega$ while R_2 decreases by $\Delta R_2 = -0.3\Omega$, use the linear approximation to estimate the change in the net resistance ΔR. Be careful to get the sign correct.

Enter exact numbers. For example, $\frac{-0.5}{7}$ should be entered as -0.5/7, NO SPACES, NO UNITS.

Solution: $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $\quad R = \boxed{\frac{4}{3}}\Omega$

$$-\frac{1}{R^2}\Delta R = -\frac{1}{R_1{}^2}\Delta R_1 - \frac{1}{R_2{}^2}\Delta R_2$$

$$\Delta R = \frac{R^2}{R_1{}^2}\Delta R_1 + \frac{R^2}{R_2{}^2}\Delta R_2 = \frac{16}{9}\frac{1}{4}(0.3) + \frac{16}{9}\frac{1}{16}(-0.3) = \frac{4}{9}0.3 - \frac{1}{9}0.3 = \frac{0.9}{9} \qquad \Delta R = \boxed{0.1}\Omega$$

9. A cardboard box has length $L = 5\,cm$, width $W = 4\,cm$ and height $H = 3\,cm$. If the length is increasing at $\frac{dL}{dt} = 0.6\frac{cm}{\sec}$ and the width is decreasing at $\frac{dW}{dt} = -0.3\frac{cm}{\sec}$, while the **surface area** is held constant, find the rate that the height is changing, $\frac{dH}{dt}$. Be careful to get the sign correct.

HINT: What is the formula for the surface area?

Solution: $A = 2LW + 2LH + 2WH$

$$\frac{dA}{dt} = \frac{\partial A}{\partial L}\frac{dL}{dt} + \frac{\partial A}{\partial W}\frac{dW}{dt} + \frac{\partial A}{\partial H}\frac{dH}{dt} = (2W+2H)\frac{dL}{dt} + (2L+2H)\frac{dW}{dt} + (2L+2W)\frac{dH}{dt}$$

$$0 = 14(0.6) + 16(-0.3) + 18\frac{dH}{dt} = 3.6 + 18\frac{dH}{dt} \qquad \frac{dH}{dt} \approx \boxed{-0.2}\frac{cm}{\sec}$$

10. The point $(-2,4)$ is a critical point of the function $g(x,y) = xy^3 - 16x^2y - 80xy$. Apply the 2nd-Derivative Test to classify $(-2,4)$.

a. Local Minimum

b. Local Maximum Correct Choice

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $g_x = y^3 - 32xy - 80y$ $\qquad g_y = 3xy^2 - 16x^2 - 80x$

Check critical point: $g_x(-2,4) = 64 + 256 - 320 = 0$ $\qquad g_y(-2,4) = -96 - 64 + 160 = 0$

$g_{xx} = -32y$ $\qquad g_{yy} = 6xy$ $\qquad g_{xy} = 3y^2 - 32x - 80$

$g_{xx}(-2,4) = -128$ $\qquad g_{yy}(-2,4) = -48$ $\qquad g_{xy}(-2,4) = 48 + 64 - 80 = 32$

$D = g_{xx}g_{yy} - g_{xy}{}^2 = 128 \cdot 48 - 32^2 = 5120$

$D > 0$ and $g_{xx} < 0$ So this is a local maximum.

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (12 points) Find all 1^{st} and 2^{nd} partial derivatives of $f(x,y) = y^2 \sin(xy)$.
Enter f_{xx} here but put all of them on your paper.

Solution: $\boxed{f_x = y^3 \cos(xy)}$ $\boxed{f_y = 2y\sin(xy) + xy^2 \cos(xy)}$

$\boxed{f_{xx} = -y^4 \sin(xy)}$ $\boxed{f_{xy} = 3y^2 \cos(xy) - xy^3 \sin(xy)}$

$\boxed{f_{yx} = 3y^2 \cos(xy) - xy^3 \sin(xy)}$ $\boxed{f_{yy} = 2\sin(xy) + 4xy\cos(xy) - x^2y^2 \sin(xy)}$

12. (18 points) The ideal gas law says the pressure, P, the density, δ, and the temperature, T, are related by $P = k\delta T$ where k is a constant. A weather balloon measures that at its current position,

$$P = .81\,atm \qquad \delta = 1.5\,\frac{kg}{m^3} \qquad T = 270°K$$

The weather balloon also measures that gradients of the density and temperature are

$$\vec{\nabla}\delta = \left\langle \frac{\partial\delta}{\partial x}, \frac{\partial\delta}{\partial y}, \frac{\partial\delta}{\partial z} \right\rangle = \langle -.2, .2, -.1 \rangle \frac{kg/m^3}{m}$$

$$\vec{\nabla}T = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle = \langle -3, 4, -12 \rangle \frac{°K}{m}$$

a. (2 pts) Find the constant k.

Solution: $k = \dfrac{P}{\delta T} = \dfrac{.81}{1.5 \cdot 270} = \dfrac{.01}{.5 \cdot 10} = \boxed{.002}$

b. (9 pts) Find the gradient of the pressure.
HINT: Find each component separately. No need to simplify numbers.
Enter $\dfrac{\partial P}{\partial y}$ here but put all three components on your paper.

Solution:

$$\frac{\partial P}{\partial x} = kT\frac{\partial\delta}{\partial x} + k\delta\frac{\partial T}{\partial x} = .002(270 \cdot (-.2) + 1.5 \cdot (-3)) = -0.117$$

$$\frac{\partial P}{\partial y} = kT\frac{\partial\delta}{\partial y} + k\delta\frac{\partial T}{\partial y} = .002(270 \cdot .2 + 1.5 \cdot 4) = 0.12$$

$$\frac{\partial P}{\partial z} = kT\frac{\partial\delta}{\partial z} + k\delta\frac{\partial T}{\partial z} = .002(270 \cdot (-.1) + 1.5 \cdot (-12)) = -0.09$$

$$\boxed{\vec{\nabla}P = \langle -0.117, 0.12, -0.09 \rangle}$$

c. (3 pts) In the absence of wind or other forces, a balloon will tend to drift from regions of high density to regions of low density. In what unit vector direction, $\hat{u}$, will this balloon drift?

Solution: The direction from high density to low density is $\vec{u} = -\vec{\nabla}\delta = \langle .2, -.2, .1 \rangle$.

$|\vec{u}| = \sqrt{(.2)^2 + (.2)^2 + (.1)^2} = \sqrt{.04 + .04 + .01} = \sqrt{.09} = .3$

$\hat{u} = \dfrac{\vec{u}}{|\vec{u}|} = \dfrac{1}{.3}\langle .2, -.2, .1 \rangle = \boxed{\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right\rangle}$

d. (4 pts) If the balloon's current velocity is $\vec{v} = \langle 4, 1, 3 \rangle \frac{m}{\sec}$, find the rate the temperature is changing as seen by the balloon.

Solution: $\dfrac{dT}{dt} = \vec{v} \cdot \vec{\nabla}T = \langle 4, 1, 3 \rangle \cdot \langle -3, 4, -12 \rangle = -12 + 4 - 36 = \boxed{-44\,\frac{°K}{\sec}}$

13. (20 points + 10 points extra credit) An aquarium has a marble base, a glass front and aluminum sides and back. There is no top. The marble costs $\$.50$ per in^2. The glass costs $\$.30$ per in^2. The aluminum costs $\$.10$ per in^2. Find the dimensions and cost of the aquarium with minimum cost if the volume needs to be $V = 5000\ in^3$. Let x be the width of the front, left to right. Let y be the width of each side, front to back. Let z be the height, top to bottom.

HINT: Work in cents.

NOTE: Solve by either Eliminating a Variable or by Lagrange Multipliers. Extra Credit for solving by both methods. Draw a line across your paper to clearly separate the two solutions.

Solution by Eliminating a Variable: Minimize the cost, written in cents:

$$C = 50xy + 30xz + 10yz + 10yz + 10xz = 50xy + 40xz + 20yz$$

subject to the constraint $V = xyz = 5000$. We solve the constraint: $z = \dfrac{5000}{xy}$ and plug into the cost:

$$C = 50xy + 40x\frac{5000}{xy} + 20y\frac{5000}{xy} = 50xy + \frac{200,000}{y} + \frac{100,000}{x}$$

We set the partial derivatives equal to zero and solve:

$$C_x = 50y - \frac{100,000}{x^2} = 0 \qquad \Rightarrow \qquad y = \frac{2000}{x^2}$$

$$C_y = 50x - \frac{200,000}{y^2} = 0 \qquad \Rightarrow \qquad x = \frac{4000}{y^2}$$

We substitute the first into the second:

$$x = 4000\left(\frac{x^2}{2000}\right)^2 = \frac{x^4}{1000} \qquad \Rightarrow \qquad x^3 = 1000 \qquad \Rightarrow \qquad x = 10$$

$$\Rightarrow \qquad y = \frac{2000}{x^2} = 20 \qquad \Rightarrow \qquad z = \frac{5000}{xy} = \frac{5000}{200} = 25$$

Then the cost in cents is:

$$C = 50xy + 40xz + 20yz = 50(10)(20) + 40(10)(25) + 20(20)(25) = 30\,000$$

So the dimensions are $\boxed{(x,y,z) = (10,20,25)}$ and the cost is $\boxed{C = \$300}$.

Solution by Lagrange Multipliers: Minimize the cost, written in cents:

$$C = 50xy + 30xz + 10yz + 10yz + 10xz = 50xy + 40xz + 20yz$$

subject to the constraint $V = xyz = 5000$. The gradients are:

$$\nabla C = \left(50y + 40z,\ 50x + 20z, 40x + 20y\right) \qquad \nabla V = (yz, xz, xy)$$

The Lagrange equations are $\nabla C = \lambda \nabla V$ or

$$50y + 40z = \lambda yz \qquad 50x + 20z = \lambda xz \qquad 40x + 20y = \lambda xy$$

If we multiply the 1^{st} equation by x, the 2^{nd} by y and the 3^{rd} by z, the right sides all become the same. So we can equate the left sides.

$$\lambda xyz = \qquad 50xy + 40xz = 50xy + 20yz = 40xz + 20yz$$

Equating the 1^{st} and 2^{nd} and then the 2^{nd} and 3^{rd}, we get:

$$50xy + 40xz = 50xy + 20yz \quad \Rightarrow \quad 40xz = 20yz \quad \Rightarrow \quad 40x = 20y \quad \Rightarrow \quad x = \frac{1}{2}y$$

$$50xy + 20yz = 40xz + 20yz \quad \Rightarrow \quad 50xy = 40xz \quad \Rightarrow \quad 50y = 40z \quad \Rightarrow \quad z = \frac{5}{4}y$$

Then the constraint $xyz = 5000$ implies: $\frac{1}{2}yy\frac{5}{4}y = 5000$ or $y^3 = 8000$ or $y = 20$.

So $x = \frac{1}{2}y = 10$ and $z = \frac{5}{4}y = 25$. Then the cost in cents is:

$$C = 50xy + 40xz + 20yz = 50(10)(20) + 40(10)(25) + 20(20)(25) = 30\,000$$

So the dimensions are $\boxed{(x,y,z) = (10,20,25)}$ and the cost is $\boxed{C = \$300}$.