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9	/20	11	/20
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MATH 251 Exam 3 Version A Fall 2020

Sections 517 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Compute $\int_0^2 \int_{x^2}^{2x} 2xy\,dy\,dx$.

a. $-\frac{16}{5}$

b. $\frac{32}{5}$

c. $\frac{16}{5}$

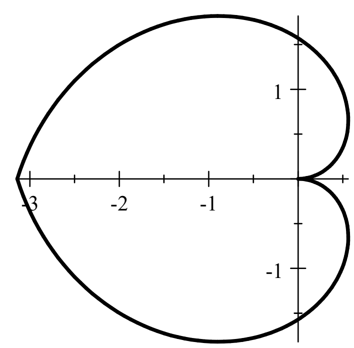
d. $\frac{32}{3}$

e. $\frac{16}{3}$ Correct Choice

Solution: $\int_0^2 \int_{x^2}^{2x} 2xy\,dy\,dx = \int_0^2 x\left[y^2\right]_{y=x^2}^{2x} dx = \int_0^2 x(4x^2 - x^4)\,dx = \int_0^2 (4x^3 - x^5)\,dx$

$= \left[x^4 - \frac{x^6}{6}\right]_0^2 = 2^4 - \frac{2^5}{3} = \boxed{\frac{16}{3}}$

2. Find the area of the heart shaped region inside the polar curve $r = |\theta|$.
HINT: Double the upper half.



a. $\frac{\pi^3}{6}$

b. $\frac{\pi^3}{3}$ Correct Choice

c. $\frac{4\pi^3}{3}$

d. $\frac{8\pi^3}{3}$

e. $\frac{16\pi^3}{3}$

Solution: For the upper half, $0 \le \theta \le \pi$ and $0 \le r \le \theta$. We double the area:

$A = 2\iint 1\,dA = 2\int_0^\pi \int_0^\theta r\,dr\,d\theta = 2\int_0^\pi \left[\frac{r^2}{2}\right]_{r=0}^{\theta} d\theta = \int_0^\pi \theta^2\,d\theta = \left[\frac{\theta^3}{3}\right]_{\theta=0}^{\pi} = \boxed{\frac{\pi^3}{3}}$

3. Find the mass of a triangular plate with vertices $(0,0)$, $(3,6)$ and $(3,-6)$ whose surface mass density is $\delta = x$.

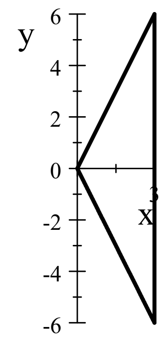
a. 12

b. 24

c. 36 Correct Choice

d. 48

e. 60



Solution: $M = \iint \delta\, dA = \int_0^3 \int_{-2x}^{2x} x\, dy\, dx = \int_0^3 \Big[xy \Big]_{y=-2x}^{2x} dx = \int_0^3 4x^2\, dx = \frac{4x^3}{3}\Big|_0^3 = \boxed{36}$

4. Find the center of mass of a triangular plate with vertices $(0,0)$, $(3,6)$ and $(3,-6)$ whose surface mass density is $\delta = x$.

a. $\left(\frac{9}{4}, 0\right)$ Correct Choice

b. $\left(0, \frac{9}{4}\right)$

c. $\left(\frac{4}{9}, 0\right)$

d. $\left(0, \frac{4}{9}\right)$

e. $(81, 0)$

Solution: $\bar{y} = 0$ by symmetry.

$M_x = \iint x\delta\, dA = \int_0^3 \int_{-2x}^{2x} x^2\, dy\, dx = \int_0^3 \Big[x^2 y \Big]_{y=-2x}^{2x} dx = \int_0^3 4x^3\, dx = x^4\Big|_0^3 = 81 \qquad \bar{x} = \frac{M_x}{M} = \frac{81}{36} = \boxed{\frac{9}{4}}$

5. Compute $\iiint x^2 + y^2\, dV$ over the region between the cones $z = \sqrt{x^2 + y^2}$ and $z = 4 - \sqrt{x^2 + y^2}$.

a. $\frac{8\pi}{3}$

b. $\frac{16\pi}{3}$

c. $\frac{32\pi}{3}$

d. $\frac{16\pi}{5}$

e. $\frac{32\pi}{5}$ Correct Choice

Solution: In cylindrical coordinates, the cones are $z = r$ and $z = 4 - r$ which intersect at $r = 2$.

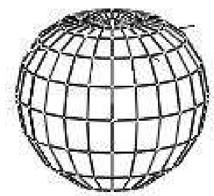
$$\int_0^{2\pi} \int_0^2 \int_r^{4-r} r^2\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 \Big[r^3 z \Big]_{z=r}^{4-r} dr = 2\pi \int_0^2 r^3(4 - 2r)\, dr = 2\pi \Big[r^4 - 2\frac{r^5}{5} \Big]_{r=0}^2$$

$$= 2\pi\left(16 - \frac{64}{5}\right) = \boxed{\frac{32\pi}{5}}$$

6. The surface of an apple is given in spherical coordinates by

$$\rho = 2 - 2\cos\varphi$$

Its volume is given by the integral:



a. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\varphi} 1\,d\rho\,d\varphi\,d\theta$

b. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{2-2\cos\varphi} 1\,d\rho\,d\varphi\,d\theta$

c. $V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1-\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

d. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{2-2\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ Correct Choice

e. $V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1}(2-2\cos\varphi)\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$

Solution: θ goes around a circle $0\cdots 2\pi$. ϕ goes North pole to South pole $0\cdots\pi$.
ρ goes from the center of the apple to the surface $0\cdots 2-2\cos\varphi$. The Jacobian is $dV = \rho^2\sin\varphi$

$$V = \iiint dV = \boxed{\int_0^{2\pi}\int_0^{\pi}\int_0^{2-2\cos\varphi} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$$

7. If $f = x^2 + y^2 - 2z^2$ and $\vec{F} = (xz, yz, -z^2)$, which of the following is FALSE?

a. $\vec{\nabla}\times\vec{\nabla}f = \vec{0}$

b. $\vec{\nabla}\cdot\vec{\nabla}f = 0$

c. $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$

d. $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{F}\right) = \vec{0}$

e. None of the above. They are all true. Correct Choice

Solution: $\vec{\nabla}\times\vec{\nabla}f = \vec{0}$ and $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$ are always true.

$\vec{\nabla}f = (2x, 2y, -4z)$ $\quad \vec{\nabla}\cdot\vec{\nabla}f = 2 + 2 - 4 = 0$

$\vec{\nabla}\cdot\vec{F} = z + z - 2z = 0$ $\quad \vec{\nabla}\left(\vec{\nabla}\cdot\vec{F}\right) = \vec{\nabla}(0) = \vec{0}$

8. Let f be a scalar potential for $\vec{F} = (y, x, z)$. Find $f(2,2,2) - f(0,0,0)$

a. 2
b. 4
c. 6 Correct Choice
d. 8
e. 10

Solution: $\vec{\nabla}f = \vec{F}$ or (1) $\partial_x f = y$ (2) $\partial_y f = x$ (3) $\partial_z f = z$

(1) $\Rightarrow$ $f = xy + g(y,z)$ $\Rightarrow$ (4) $\partial_y f = x + \partial_y g$

(2) and (4) $\Rightarrow$ $\partial_y g = 0$ $\Rightarrow$ $g = h(z)$ $\Rightarrow$ $f = xy + h(z)$ $\Rightarrow$ (5) $\partial_z f = \dfrac{dh}{dz}$

(3) and (5) $\Rightarrow$ $\dfrac{dh(z)}{dz} = z$ $\Rightarrow$ $h = \dfrac{z^2}{2} + C$ $\Rightarrow$ $f = xy + \dfrac{z^2}{2} + C$

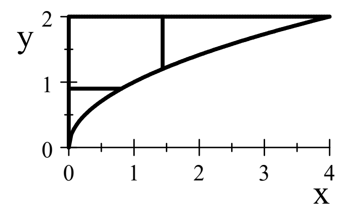
It can also be done by inspection.

So $f(2,2,2) - f(0,0,0) = 2\cdot 2 + \dfrac{2^2}{2} - 0 = \boxed{6}$.

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (20 points) Draw the region of integration and compute $\int_0^4 \int_{\sqrt{x}}^2 \sqrt{y^3+1}\,dy\,dx$ by reversing the order of integration.

Solution: To reverse the order of integration plot the region $0 \le x \le 4,\quad \sqrt{x} \le y \le 2.$
Include a vertical line to indicate the y limits.
Add a horizontal line to indicate the new x limits.
The new limits are $0 \le y \le 2,\quad 0 \le x \le y^2.$
We write the new integral and compute it.



$$\int_0^2 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^2 \sqrt{y^3+1}\,[x]_0^{y^2}\,dy = \int_0^2 \sqrt{y^3+1}\,y^2\,dy = \frac{2}{9}(y^3+1)^{3/2}\Big|_0^2 = \frac{2}{9}(9^{3/2}-1) = \boxed{\frac{52}{9}}$$

10. (20 points) Consider the vector field $\vec{F} = (x^3, y^3, x^2z + y^2z)$. First compute $\vec{\nabla}\cdot\vec{F}$ in rectangular coordinates. Then convert $\vec{\nabla}\cdot\vec{F}$ into cylindrical coordinates. Finally, compute $\iiint \vec{\nabla}\cdot\vec{F}\,dV$ over the solid region below the cone $z = 4 - \sqrt{x^2+y^2}$ and above the xy-plane.

Solution: $\vec{\nabla}\cdot\vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = \boxed{4(x^2+y^2)} = \boxed{4r^2}$

$$z = 4 - \sqrt{x^2+y^2} = 4 - r = 0 \qquad r = 4$$

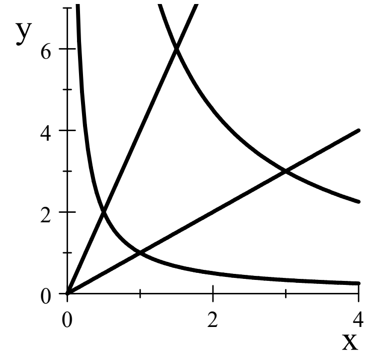
$$\iiint \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^4\int_0^{4-r} 4r^2\,r\,dz\,dr\,d\theta = 2\pi\int_0^4 \left[4r^3 z\right]_{z=0}^{4-r} dr = 2\pi\int_0^4 4r^3(4-r)\,dr$$

$$= 8\pi\left[r^4 - \frac{r^5}{5}\right]_0^4 = 8\pi 4^4\left(1-\frac{4}{5}\right) = \boxed{\frac{2048\pi}{5}}$$

11. (20 points) Compute $\iint_R x^2 \, dx\, dy$ over the diamond shaped region R bounded by

$xy = 1, \quad xy = 9, \quad y = x, \quad y = 4x$

HINT: Use the curvilinear coordinates (u, v) where $x = uv$ and $y = \frac{u}{v}$.



a. (4 pts) What are the boundaries in terms of u and v?

Solution: $xy = uv\frac{u}{v} = u^2 = 1$ or 9. So $\boxed{u = 1}$ and $\boxed{u = 3}$ are the u-boundaries.
$\frac{y}{x} = \frac{u}{vuv} = \frac{1}{v^2} = 1$ or 4. So $\boxed{v = 1}$ and $\boxed{v = \frac{1}{2}}$ are the v-boundaries.

b. (6 pts) Find the Jacobian factor $J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$.

Solution: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & \frac{1}{v} \\ u & -\frac{u}{v^2} \end{vmatrix} == -v\frac{u}{v^2} - \frac{u}{v} = -\frac{2u}{v}$

$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \boxed{\frac{2u}{v}}$ because u and v are positive.

c. (3 pts) Express the integrand, x^2, in terms of u and v.

Solution: $x^2 = \boxed{u^2 v^2}$

d. (7 pts) Compute the integral.

Solution:
$$\iint_R x^2 \, dA = \iint_R x^2 J \, du\, dv = \int_{1/2}^{1} \int_{1}^{3} u^2 v^2 \frac{2u}{v} \, du\, dv = 2\int_{1/2}^{1} \int_{1}^{3} u^3 v \, du\, dv$$
$$= 2\left[\frac{u^4}{4}\right]_{u=1}^{3} \left[\frac{v^2}{2}\right]_{v=1/2}^{1} = 2\left[\frac{81-1}{4}\right]\left[\frac{1}{2} - \frac{1}{8}\right] = 2(20)\left(\frac{3}{8}\right) = \boxed{15}$$