

Name \_\_\_\_\_

MATH 251 Exam 3 Version B Fall 2020

Sections 519 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-8	/48	10	/20
9	/20	11	/20
		Total	/108

1. Compute  $\int_0^2 \int_{x^2}^{2x} 2xy \, dy \, dx$ .

a.  $-\frac{16}{5}$

b.  $\frac{32}{5}$

c.  $\frac{16}{5}$

d.  $\frac{32}{3}$

e.  $\frac{16}{3}$  Correct Choice

**Solution:**  $\int_0^2 \int_{x^2}^{2x} 2xy \, dy \, dx = \int_0^2 x \left[ y^2 \right]_{y=x^2}^{2x} dx = \int_0^2 x(4x^2 - x^4) dx = \int_0^2 (4x^3 - x^5) dx$   
 $= \left[ x^4 - \frac{x^6}{6} \right]_0^2 = 2^4 - \frac{2^6}{6} = \frac{16}{3}$

2. Find the area of the heart shaped region inside the polar curve  $r = |\theta|$ .

HINT: Double the upper half.

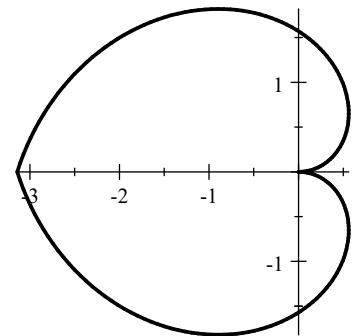
a.  $\frac{\pi^3}{6}$

b.  $\frac{\pi^3}{3}$  Correct Choice

c.  $\frac{4\pi^3}{3}$

d.  $\frac{8\pi^3}{3}$

e.  $\frac{16\pi^3}{3}$



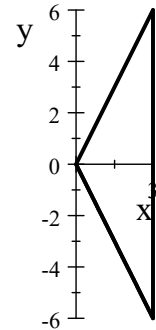
**Solution:** For the upper half,  $0 \leq \theta \leq \pi$  and  $0 \leq r \leq \theta$ . We double the area:

$$A = 2 \iint 1 \, dA = 2 \int_0^\pi \int_0^\theta r \, dr \, d\theta = 2 \int_0^\pi \left[ \frac{r^2}{2} \right]_{r=0}^\theta d\theta = \int_0^\pi \theta^2 \, d\theta = \left[ \frac{\theta^3}{3} \right]_{\theta=0}^\pi = \frac{\pi^3}{3}$$

3. Find the mass of a triangular plate with vertices  $(0,0)$ ,  $(3,6)$  and  $(3,-6)$  whose surface mass density is  $\delta = x$ .

- a. 12
- b. 24
- c. 36 Correct Choice
- d. 48
- e. 60

**Solution:** 
$$M = \iint \delta dA = \int_0^3 \int_{-2x}^{2x} x dy dx = \int_0^3 [xy]_{y=-2x}^{2x} dx = \int_0^3 4x^2 dx = \frac{4x^3}{3} \Big|_0^3 = \boxed{36}$$



4. Find the center of mass of a triangular plate with vertices  $(0,0)$ ,  $(3,6)$  and  $(3,-6)$  whose surface mass density is  $\delta = x$ .

- a.  $(\frac{9}{4}, 0)$  Correct Choice
- b.  $(0, \frac{9}{4})$
- c.  $(\frac{4}{9}, 0)$
- d.  $(0, \frac{4}{9})$
- e.  $(81, 0)$

**Solution:**  $\bar{y} = 0$  by symmetry.

$$M_x = \iint x\delta dA = \int_0^3 \int_{-2x}^{2x} x^2 dy dx = \int_0^3 [x^2y]_{y=-2x}^{2x} dx = \int_0^3 4x^3 dx = x^4 \Big|_0^3 = 81 \quad \bar{x} = \frac{M_x}{M} = \frac{81}{36} = \boxed{\frac{9}{4}}$$

5. Compute  $\iiint x^2 + y^2 dV$  over the region between the cones  $z = \sqrt{x^2 + y^2}$  and  $z = 4 - \sqrt{x^2 + y^2}$ .

- a.  $\frac{8\pi}{3}$
- b.  $\frac{16\pi}{3}$
- c.  $\frac{32\pi}{3}$
- d.  $\frac{16\pi}{5}$
- e.  $\frac{32\pi}{5}$  Correct Choice

**Solution:** In cylindrical coordinates, the cones are  $z = r$  and  $z = 4 - r$  which intersect at  $r = 2$ .

$$\begin{aligned} \int_0^{2\pi} \int_0^2 \int_r^{4-r} r^2 r dz dr d\theta &= 2\pi \int_0^2 [r^3 z]_{z=r}^{4-r} dr = 2\pi \int_0^2 r^3 (4 - 2r) dr = 2\pi \left[ r^4 - 2\frac{r^5}{5} \right]_{r=0}^2 \\ &= 2\pi \left( 16 - \frac{64}{5} \right) = \boxed{\frac{32\pi}{5}} \end{aligned}$$

6. The surface of an apple is given in spherical coordinates by

$$\rho = 2 - 2 \cos \phi$$

Its volume is given by the integral:

a.  $V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1-\cos\phi} 1 \, d\rho \, d\phi \, d\theta$

b.  $V = \int_0^{2\pi} \int_0^{\pi} \int_0^{2-2\cos\phi} 1 \, d\rho \, d\phi \, d\theta$

c.  $V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1-\cos\phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$

d.  $V = \int_0^{2\pi} \int_0^{\pi} \int_0^{2-2\cos\phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$  Correct Choice

e.  $V = \int_0^{2\pi} \int_0^{\pi} \int_0^1 (2 - 2 \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$



**Solution:**  $\theta$  goes around a circle  $0 \dots 2\pi$ .  $\phi$  goes North pole to South pole  $0 \dots \pi$ .

$\rho$  goes from the center of the apple to the surface  $0 \dots 2 - 2 \cos \phi$ . The Jacobian is  $dV = \rho^2 \sin \phi$

$$V = \iiint dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{2-2\cos\phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

7. If  $f = x^2 + y^2 - 2z^2$  and  $\vec{F} = (xz, yz, -z^2)$ , which of the following is FALSE?

a.  $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$

b.  $\vec{\nabla} \cdot \vec{\nabla} f = 0$

c.  $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$

d.  $\vec{\nabla}(\vec{\nabla} \cdot \vec{F}) = \vec{0}$

e. None of the above. They are all true. Correct Choice

**Solution:**  $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$  and  $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$  are always true.

$$\vec{\nabla} f = (2x, 2y, -4z) \quad \vec{\nabla} \cdot \vec{\nabla} f = 2 + 2 - 4 = 0$$

$$\vec{\nabla} \cdot \vec{F} = z + z - 2z = 0 \quad \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) = \vec{\nabla}(0) = \vec{0}$$

8. Let  $f$  be a scalar potential for  $\vec{F} = (y, x, z)$ . Find  $f(2, 2, 2) - f(0, 0, 0)$

a. 2

b. 4

c. 6 Correct Choice

d. 8

e. 10

**Solution:**  $\vec{\nabla} f = \vec{F}$  or (1)  $\partial_x f = y$  (2)  $\partial_y f = x$  (3)  $\partial_z f = z$

$$(1) \Rightarrow f = xy + g(y, z) \Rightarrow (4) \partial_y f = x + \partial_y g$$

$$(2) \text{ and } (4) \Rightarrow \partial_y g = 0 \Rightarrow g = h(z) \Rightarrow f = xy + h(z) \Rightarrow (5) \partial_z f = \frac{dh}{dz}$$

$$(3) \text{ and } (5) \Rightarrow \frac{dh(z)}{dz} = z \Rightarrow h = \frac{z^2}{2} + C \Rightarrow f = xy + \frac{z^2}{2} + C$$

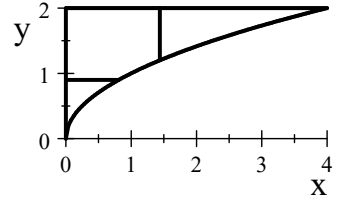
It can also be done by inspection.

$$\text{So } f(2, 2, 2) - f(0, 0, 0) = 2 \cdot 2 + \frac{2^2}{2} - 0 = \boxed{6}.$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (20 points) Draw the region of integration and compute  $\int_0^4 \int_{\sqrt{x}}^2 \sqrt{y^3 + 1} dy dx$  by reversing the order of integration.

**Solution:** To reverse the order of integration plot the region  $0 \leq x \leq 4, \sqrt{x} \leq y \leq 2$ .  
 Include a vertical line to indicate the  $y$  limits.  
 Add a horizontal line to indicate the new  $x$  limits.  
 The new limits are  $0 \leq y \leq 2, 0 \leq x \leq y^2$ .  
 We write the new integral and compute it.



$$\int_0^2 \int_0^{y^2} \sqrt{y^3 + 1} dx dy = \int_0^2 \sqrt{y^3 + 1} [x]_0^{y^2} dy = \int_0^2 \sqrt{y^3 + 1} y^2 dy = \frac{2}{9} (y^3 + 1)^{3/2} \Big|_0^2 = \frac{2}{9} (9^{3/2} - 1) = \boxed{\frac{52}{9}}$$

10. (20 points) Consider the vector field  $\vec{F} = (x^3, y^3, x^2z + y^2z)$ . First compute  $\vec{\nabla} \cdot \vec{F}$  in rectangular coordinates. Then convert  $\vec{\nabla} \cdot \vec{F}$  into cylindrical coordinates. Finally, compute  $\iiint \vec{\nabla} \cdot \vec{F} dV$  over the solid region below the cone  $z = 4 - \sqrt{x^2 + y^2}$  and above the  $xy$ -plane.

**Solution:**  $\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = \boxed{4(x^2 + y^2)} = \boxed{4r^2}$

$$z = 4 - \sqrt{x^2 + y^2} = 4 - r = 0 \quad r = 4$$

$$\begin{aligned} \iiint \vec{\nabla} \cdot \vec{F} dV &= \int_0^{2\pi} \int_0^4 \int_0^{4-r} 4r^2 r dz dr d\theta = 2\pi \int_0^4 [4r^3 z]_{z=0}^{4-r} dr = 2\pi \int_0^4 4r^3(4-r) dr \\ &= 8\pi \left[ r^4 - \frac{r^5}{5} \right]_0^4 = 8\pi 4^4 \left( 1 - \frac{4}{5} \right) = \boxed{\frac{2048\pi}{5}} \end{aligned}$$

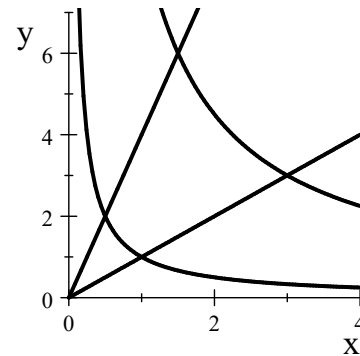
11. (20 points) Compute  $\iint_R x^2 dx dy$  over

the diamond shaped region  $R$  bounded by

$$xy = 1, \quad xy = 9, \quad y = x, \quad y = 4x$$

HINT: Use the curvilinear coordinates  $(u, v)$

where  $x = uv$  and  $y = \frac{u}{v}$ .



a. (4 pts) What are the boundaries in terms of  $u$  and  $v$ ?

**Solution:**  $xy = uv \frac{u}{v} = u^2 = 1$  or  $9$ . So  $u = 1$  and  $u = 3$  are the  $u$ -boundaries.

$\frac{y}{x} = \frac{u}{vuv} = \frac{1}{v^2} = 1$  or  $4$ . So  $v = 1$  and  $v = \frac{1}{2}$  are the  $v$ -boundaries.

b. (6 pts) Find the Jacobian factor  $J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$ .

$$\text{Solution: } \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & \frac{1}{v} \\ u & -\frac{u}{v^2} \end{vmatrix} = -v \frac{u}{v^2} - \frac{u}{v} = -\frac{2u}{v}$$

$$J = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{2u}{v} \text{ because } u \text{ and } v \text{ are positive.}$$

c. (3 pts) Express the integrand,  $x^2$ , in terms of  $u$  and  $v$ .

$$\text{Solution: } x^2 = u^2 v^2$$

d. (7 pts) Compute the integral.

**Solution:**

$$\begin{aligned} \iint_R x^2 dA &= \iint_R x^2 J du dv = \int_{1/2}^1 \int_1^3 u^2 v^2 \frac{2u}{v} du dv = 2 \int_{1/2}^1 \int_1^3 u^3 v du dv \\ &= 2 \left[ \frac{u^4}{4} \right]_{u=1}^3 \left[ \frac{v^2}{2} \right]_{v=1/2}^1 = 2 \left[ \frac{81-1}{4} \right] \left[ \frac{1}{2} - \frac{1}{8} \right] = 2(20) \left( \frac{3}{8} \right) = 15 \end{aligned}$$