Name________________________

1-9	/45	11	/20
10	/20	12	/20
		Total	/105

MATH 251 Final Exam Version A Fall 2020

Sections 517/519 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^2 \int_0^z \int_0^{xz} 30x\,dy\,dx\,dz$.

a. 4

b. 8

c. 16

d. 32

e. 64 Correct Choice

$$\int_0^2 \int_0^z \int_0^{xz} 30x\,dy\,dx\,dz = \int_0^2 \int_0^z \left[30xy\right]_{y=0}^{xz} dx\,dz = \int_0^2 \int_0^z 30x^2 z\,dx\,dz = \int_0^2 \left[10x^3 z\right]_{x=0}^{z} dz = \int_0^2 10z^4\,dz$$

$$= \left[2z^5\right]_{z=0}^{2} = 64$$

2. Find the center of mass of the quarter circle $x^2 + y^2 \le 9$ in the first quadrant, if the density is $\delta = \sqrt{x^2 + y^2}$.

a. $(\bar{x},\bar{y}) = \left(\frac{9}{4}, \frac{9}{4}\right)$

b. $(\bar{x},\bar{y}) = \left(\frac{9}{2}, \frac{9}{2}\right)$

c. $(\bar{x},\bar{y}) = \left(\frac{2}{9}, \frac{2}{9}\right)$

d. $(\bar{x},\bar{y}) = \left(\frac{9}{2\pi}, \frac{9}{2\pi}\right)$ Correct Choice

e. $(\bar{x},\bar{y}) = \left(\frac{2\pi}{9}, \frac{2\pi}{9}\right)$

Solution: $M = \iint \delta\,dA = \int_0^{\pi/2} \int_0^3 r\,r\,dr\,d\theta = \frac{\pi}{2}\left[\frac{r^3}{3}\right]_0^3 = \frac{9\pi}{2}$ $\bar{x} = \bar{y}$ by symmetry

$M_x = \iint x\delta\,dA = \int_0^{\pi/2} \int_0^3 r\cos(\theta) r\,r\,dr\,d\theta = [\sin(\theta)]_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^3 = \frac{81}{4}$ $\bar{x} = \frac{M_x}{M} = \frac{81}{4}\frac{2}{9\pi} = \frac{9}{2\pi}$

3. The temperature in an ideal gas is given by $T = \kappa \frac{P}{\delta}$ where κ is a constant, P is the pressure and δ is the density. At a certain point $Q = (3,2,1)$, we have

$$P(Q) = 8 \qquad \vec{\nabla} P(Q) = (4,-2,-4)$$
$$\delta(Q) = 2 \qquad \vec{\nabla}\delta(Q) = (-1,4,2)$$

So at the point Q, the temperature is $T(Q) = 4\kappa$ and its gradient is $\vec{\nabla} T(Q) =$

a. $\kappa(-8.5,6,9)$

b. $\kappa(4,-9,-6)$ Correct Choice

c. $\kappa(3,2,-2)$

d. $\kappa\left(\frac{1}{2},2\right)$

e. $\kappa\left(-\frac{1}{2},2\right)$

Solution: By chain rule: (Think about each component separately.)

$$\vec{\nabla} T = \frac{\partial T}{\partial P}\vec{\nabla} P + \frac{\partial T}{\partial \delta}\vec{\nabla}\delta = \frac{\kappa}{\delta}\vec{\nabla} P - \frac{\kappa P}{\delta^2}\vec{\nabla}\delta = \frac{\kappa}{2}(4,-2,-4) - \frac{\kappa 8}{2^2}(-1,4,2)$$
$$= \kappa(2,-1,-2) + \kappa(2,-8,-4) = \kappa(4,-9,-6)$$

4. Compute $\iint\limits_C e^{-x^2-y^2}\,dx\,dy$ over the disk enclosed in the circle $x^2 + y^2 = 4$.

a. $\frac{\pi}{2}(1 - e^{-4})$

b. $\pi(1 - e^{-4})$ Correct Choice

c. $\frac{\pi}{2}e^{-4}$

d. πe^{-4}

e. $2\pi e^{-4}$

Solution: $\iint e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^2 = \pi(1 - e^{-4})$

5. Find the volume below $z = xy$ above the region between the curves $y = 3x$ and $y = x^2$.

a. $\frac{81}{2}$

b. $\frac{81}{4}$

c. $\frac{81}{8}$

d. $\frac{243}{2}$

e. $\frac{243}{8}$ Correct Choice

Solution: $3x = x^2 \quad \Rightarrow \quad x = 0,3$

$$V = \int_0^3 \int_{x^2}^{3x} xy\,dy\,dx = \int_0^3 \left[\frac{xy^2}{2} \right]_{y=x^2}^{3x} dx = \int_0^3 \left(\frac{x9x^2}{2} - \frac{xx^4}{2} \right) dx = \left[\frac{9x^4}{8} - \frac{x^6}{12} \right]_{x=0}^{3}$$

$$= \frac{3^6}{4}\left(\frac{1}{2} - \frac{1}{3} \right) = \frac{243}{8}$$

6. Compute the line integral $\int_P \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = \langle y,x \rangle$ along the parabola $y = x^2$ from $x = -1$ to $x = 2$.

HINT: Find a scalar potential.

a. 9 Correct Choice

b. 7

c. 5

d. 3

e. 1

Solution: We find a scalar potential. $\vec{F} = \vec{\nabla} f \qquad \partial_x f = y \quad \partial_y f = x \quad \Rightarrow \quad f(x,y) = xy$

The endpoints are $(-1,1)$ and $(2,4)$.

By the FTCC, $\int_P \vec{\nabla} f \cdot ds = f(2,4) - f(-1,1) = 8 - -1 = 9$

7. Compute $\iint \frac{1}{y}\,dS$ on the parametric surface $\vec{R}(u,v) = (u^2 - v^2, u^2 + v^2, 2uv)$

for $1 \le u \le 3$ and $1 \le v \le 4$.
HINT: Find the normal vector.

a. $6\sqrt{2}$

b. $12\sqrt{2}$

c. $24\sqrt{2}$ Correct Choice

d. $64\sqrt{2}$

e. $272\sqrt{2}$

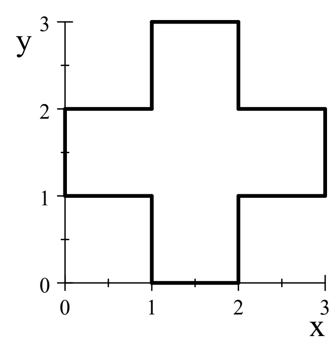
Solution:
$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (2u & 2u & 2v) \\ \vec{e}_v = & (-2v & 2v & 2u) \end{array} \qquad \vec{N} = \vec{e}_u \times \vec{e}_v = (4u^2 - 4v^2, -4v^2 - 4u^2, 8uv)$$

$$\left|\vec{N}\right| = \sqrt{(4u^2 - 4v^2)^2 + (-4v^2 - 4u^2)^2 + (-8uv)^2} = \sqrt{32u^4 + 64u^2v^2 + 32v^4}$$
$$= \sqrt{32(u^4 + 2u^2v^2 + v^4)} = 4\sqrt{2}\,(u^2 + v^2)$$
$$\frac{1}{y} = \frac{1}{u^2 + v^2}$$
$$\iint \frac{1}{y}\,dS = \int_1^4 \int_1^3 \frac{1}{u^2 + v^2} 4\sqrt{2}\,(u^2 + v^2)\,du\,dv = 4\sqrt{2} \int_1^4 \int_1^3 1\,du\,dv = 24\sqrt{2}$$

8. Compute $\oint (2x\sin y - 5y)\,dx + (x^2\cos y - 4x)\,dy$

counterclockwise around the cross shown.

HINT: Use Green's Theorem.



a. -45

b. -10

c. 5 Correct Choice

d. 10

e. 45

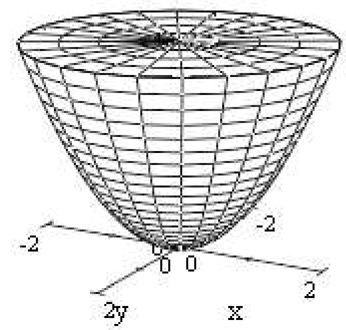
Solution: Green's Theorem says $\oint_{\partial R} P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$

Here $P = 2x\sin y - 5y$ and $Q = x^2\cos y - 4x$.

So $\partial_x Q - \partial_y P = (2x\cos y - 4) - (2x\cos y - 5) = 1$.

So $\iint_R (\partial_x Q - \partial_y P)\,dx\,dy = \iint_R 1\,dx\,dy =$ area $= 5$

9. Compute $\iint \vec{F} \cdot d\vec{S}$ for $\vec{F} = (xy^2, yx^2, z(x^2+y^2))$
over the complete surface of the solid
above the paraboloid $z = x^2 + y^2$
below the plane $z = 4$, oriented outward.
Note: The paraboloid may be parametrized by
$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$
Hint: Use a Theorem.



a. $\frac{64}{5}\pi$

b. $\frac{64}{3}\pi$ Correct Choice

c. $\frac{64}{15}\pi$

d. $\frac{256}{15}\pi$

e. $\frac{256}{3}\pi$

SOLUTION: By Gauss' Theorem $\iint \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{F}\, dV$ Use cylindrical coordinates.

$\vec{\nabla} \cdot \vec{F} = y^2 + x^2 + x^2 + y^2 = 2r^2 \qquad dV = r\,dr\,d\theta\,dz$

$$\iint \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 2r^3\,dz\,dr\,d\theta = 2\pi \int_0^2 \left[2r^3 z\right]_{r^2}^4 dr = 2\pi \int_0^2 (8r^3 - 2r^5)\,dr = 2\pi\left[2r^4 - \frac{r^6}{3}\right]_0^2$$
$$= 2\pi\left(32 - \frac{64}{3}\right) = \frac{64}{3}\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (20 points) Consider the surface which is the graph of the equation $xy - z^2 = 5$.

Letter the parts. Box your answers.

a. Find the normal vector to the surface at the point $P = (3,2,1)$.

Solution: Let $f = xy - z^2$. Then $\vec{\nabla} f = \langle y, x, -2z \rangle$.

Then the normal at $P = (3,2,1)$ is $\vec{N} = \vec{\nabla} f\big|_P = \boxed{\langle 2,3,-2 \rangle}$.

b. Find the standard equation of the tangent plane to the surface at the point $P = (3,2,1)$. Then find its z-intercept.

Solution: The equation of the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $2x + 3y - 2z = 2 \cdot 3 + 3 \cdot 2 - 2 \cdot 1$ or $\boxed{2x + 3y - 2z = 10}$

Solve for $z = x + \frac{3}{2} y - 5$. So the z-intercept is $\boxed{c = -5}$.

c. Find the parametric equation of the normal line to the surface at the point $P = (3,2,1)$. Then find where the normal line intersects the xy-plane.

Solution: The equation of a line is $X = P + t\vec{v}$ where the direction $\vec{v}$ is the normal $\vec{v} = \vec{N} = \langle 2,3,-2 \rangle$.

So $(x,y,z) = (3,2,1) + t\langle 2,3,-2 \rangle$ or $(x,y,z) = (3 + 2t, 2 + 3t, 1 - 2t)$.

The line intersects the xy-plane when $z = 1 - 2t = 0$ or $t = \frac{1}{2}$.

So the intersection point is $(x,y,z) = (3 + 2t, 2 + 3t, 1 - 2t) = \boxed{\left(4, \frac{7}{2}, 0\right)}$.

11. (20 points) Find the point in the first octant on the graph of $z = \dfrac{8}{x^2 y}$ closest to the origin.

What is its distance from the origin? Box your answers.

Solution by Eliminating a Variable:

We minimize the square of the distance $f = D^2 = x^2 + y^2 + z^2$ subject to the constraint $z = \dfrac{8}{x^2 y}$.

$$f = x^2 + y^2 + \frac{64}{x^4 y^2}$$

We set the derivatives equal to 0 and solve:

$f_x = 2x - \dfrac{256}{x^5 y^2} = 0 \qquad f_y = 2y - \dfrac{128}{x^4 y^3} = 0$ or $x^6 y^2 = 128 \qquad x^4 y^4 = 64$

So $y = \dfrac{\sqrt[4]{64}}{x} = \dfrac{\sqrt{8}}{x}$ and $x^6 \dfrac{8}{x^2} = 128$ or $x^4 = 16$

So $x = 2 \qquad y = \dfrac{\sqrt{8}}{2} = \sqrt{2} \qquad z = \dfrac{8}{2^2\sqrt{2}} = \sqrt{2}$. $\boxed{(x,y,z) = \left(2, \sqrt{2}, \sqrt{2}\right)}$

$D = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+2+2} = \boxed{\sqrt{8}}$

Solution by Lagrange Multipliers:

We minimize the square of the distance $f = D^2 = x^2 + y^2 + z^2$ subject to the constraint $g = x^2 yz = 8$.

The gradients are:

$$\nabla f = (2x, 2y, 2z) \qquad \nabla g = (2xyz, x^2 z. x^2 y)$$

So the Lagrange equations are $\nabla f = \lambda \nabla g$ or

$$2x = \lambda 2xyz \qquad 2y = \lambda x^2 z \qquad 2z = \lambda x^2 y$$

We solve each for λ and equate them:

$$\lambda = \qquad \frac{1}{yz} = \frac{2y}{x^2 z} = \frac{2z}{x^2 y}$$

The second equation gives $y = z$. The first equation gives $x^2 = 2y^2$ or $x = \sqrt{2}\,y$. So the constraint becomes

$$x^2 yz = 2y^2 yy = 8 \quad \text{or} \quad y = 4^{1/4} = \sqrt{2}$$

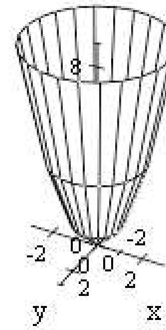
Substiting back, we find $x = \sqrt{2}\,y = 2$ and $z = y = \sqrt{2}$. $\boxed{(x,y,z) = \left(2, \sqrt{2}, \sqrt{2}\right)}$

$D = \sqrt{x^2 + y^2 + z^2} = \sqrt{4+2+2} = \boxed{\sqrt{8}}$

12. (20 points) Verify Stokes' Theorem $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial P} \vec{F} \cdot d\vec{s}$

for the vector field $\vec{F} = \langle -yz, xz, z^2 \rangle$ and the

paraboloid $z = x^2 + y^2$ for $z \le 9$ oriented down and out.



Be sure to check and explain the orientations. Use the following steps.

Letter the parts. Box your answers.

LHS:

a. Compute the curl $\vec{\nabla} \times \vec{F}$ in rectangular coordinates.

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(-x) - \hat{\jmath}(--y) + \hat{k}(z - -z) = \langle -x, -y, 2z \rangle$

b. The paraboloid surface, P, may be paranmetrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

What is $\vec{\nabla} \times \vec{F}$ on the paraboloid?

Solution: $\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = \langle -r\cos\theta, -r\sin\theta, 2r^2 \rangle$

c. Find the normal to the paraboloid.

Solution: $\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 2r) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{vmatrix}$

$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(- -2r^2\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$

$\vec{N}$ the orientation is up and in. It should be down and out. Reverse it: $\vec{N} = \boxed{\langle 2r^2\cos\theta, 2r^2\sin\theta, -r \rangle}$

d. Compute the integral $\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S}$.

Solution: $\vec{\nabla} \times \vec{F} \cdot \vec{N} = -2r^3\cos^2\theta - 2r^3\sin^2\theta - 2r^3 = -4r^3$

$$\iint_P \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_P \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi} \int_0^3 (-4r^3)\, dr\, d\theta = -2\pi[r^4]_0^3 = -2\pi 3^4 = \boxed{-162\pi}$$

RHS:

e. The circle, C, at the top of the paraboloid may be parametrized by $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 9)$.

What is $\vec{F}$ on the circle?

Solution: $\vec{F} = \langle -yz, xz, z^2 \rangle = \boxed{\langle -27\sin\theta, 27\cos\theta, 81 \rangle}$

f. What is the tangent vector to the circle?

Solution: $\vec{v} = \langle -3\sin\theta, 3\cos\theta, 0 \rangle$

In the 1^{st} quadrant, $v_1 \leq 0$ and $v_2 \geq 0$. So $\vec{v}$ points counterclockwise.
By the RHR, we need $\vec{v}$ clockwise. Reverse it: $\vec{v} = \boxed{\langle 3\sin\theta, -3\cos\theta, 0 \rangle}$

g. Compute the integral $\oint_{\partial P} \vec{F} \cdot d\vec{s}$.

$$\oint_{\partial P} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -81\sin^2\theta - 81\cos^2\theta\, d\theta = -81 \int_0^{2\pi} d\theta = -162\pi$$

They agree!