Name____________________________

1-9	/45	11	/20
10	/20	12	/20
		Total	/105

MATH 251 Final Exam Version B Fall 2020

Sections 517/519 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_1^2 \int_{1/y}^1 ye^{xy}\,dx\,dy$.

a. $e^2 - 2e$ Correct Choice

b. $e^2 - e$

c. $e^2 - 2e - 1$

d. $e^2 - e - 1$

e. $e^2 - 2$

$$\int_1^2 \int_{1/y}^1 ye^{xy}\,dx\,dy = \int_1^2 \left[e^{xy}\right]_{x=1/y}^1 dy = \int_1^2 [e^y - e]\,dy = \left[e^y - ey\right]_{y=1}^2 = e^2 - 2e$$

2. Find the center of mass of the half circle $x^2 + y^2 \le 9$ with $y \ge 0$, if the density is $\delta = \sqrt{x^2+y^2}$.

a. $(\bar{x},\bar{y}) = \left(0, \frac{9}{4}\right)$

b. $(\bar{x},\bar{y}) = \left(0, \frac{2}{9}\right)$

c. $(\bar{x},\bar{y}) = \left(0, \frac{9}{2}\right)$

d. $(\bar{x},\bar{y}) = \left(0, \frac{2\pi}{9}\right)$

e. $(\bar{x},\bar{y}) = \left(0, \frac{9}{2\pi}\right)$ Correct Choice

Solution: $M = \iint \delta\,dA = \int_0^\pi \int_0^3 r\,r\,dr\,d\theta = \pi\left[\frac{r^3}{3}\right]_0^3 = 9\pi$ $\quad \bar{x} = 0$ by symmetry

$$M_y = \iint y\delta\,dA = \int_0^{\pi/2} \int_0^3 r\sin(\theta) r\,r\,dr\,d\theta = \left[-\cos(\theta)\right]_0^\pi \left[\frac{r^4}{4}\right]_0^3 = \frac{81}{2} \qquad \bar{y} = \frac{M_y}{M} = \frac{81}{2}\frac{1}{9\pi} = \frac{9}{2\pi}$$

3. Ham Deut is flying the Millennium Eagle through a dangerous zenithon field whose density is $\rho = xyz$. If his current position is $(x,y,z) = (1,-1,2)$, in what **unit** vector direction should he travel to **decrease** the density as fast as possible?

a. $(2,-2,1)$

b. $\left(\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right)$ Correct Choice

c. $(-2,2,-1)$

d. $\left(\frac{-2}{3},\frac{2}{3},\frac{-1}{3}\right)$

e. $\left(\frac{-2}{3},\frac{-2}{3},\frac{-1}{3}\right)$

Solution: $\vec{\nabla}\rho = (yz,xz,xy)$ $\quad \vec{\nabla}\rho\Big|_{(1,-1,2)} = (-2,2,-1)$ $\quad \left|\vec{\nabla}\rho\right| = \sqrt{4+4+1} = 3$

He should travel in the direction $\vec{v} = -\vec{\nabla}\rho = (2,-2,1)$ or the unit direction $\hat{v} = \left(\frac{2}{3},\frac{-2}{3},\frac{1}{3}\right)$.

4. Compute $\int_0^8 \int_{x^{1/3}}^2 \cos(y^2)\,dy\,dx$

HINT: Reverse the order of integration.

a. $\frac{1}{4}\sin(4) - \frac{1}{4}$

b. $\frac{1}{4}\sin(16) - \frac{1}{4}$

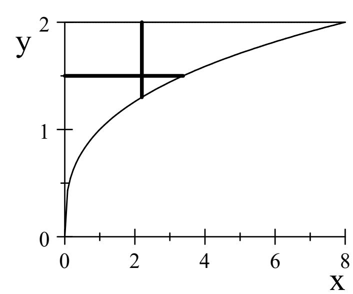
c. $\frac{1}{4}\sin(16)$ Correct Choice

d. $\frac{1}{4}\sin(64) - \frac{1}{4}$

e. $\frac{1}{4}\sin(64)$

Solution: Plot the region. Reverse the order.

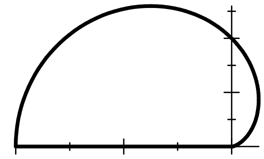
Compute new limits: $y = x^{1/3} \quad \Rightarrow \quad x = y^3$



$$\int_0^8 \int_{x^{1/3}}^2 \cos(y^2)\,dy\,dx = \int_0^2 \int_0^{y^3} \cos(y^4)\,dx\,dy = \int_0^2 \cos(y^4)\big[x\big]_{x=0}^{y^3}\,dy$$

$$= \int_0^2 y^3 \cos(y^4)\,dy = \left[\frac{\sin(y^4)}{4}\right]_{y=o}^2 = \frac{1}{4}\sin(16)$$

5. Find the volume below $z = y$ above the region between the x-axis and the upper half of the cardioid $r = 1 - \cos\theta$.



a. $\frac{1}{12}$

b. $\frac{1}{6}$

c. $\frac{2}{3}$

d. $\frac{4}{3}$ Correct Choice

e. $\frac{8}{3}$

Solution: $V = \iint y\,dA = \int_0^\pi \int_0^{1-\cos\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^\pi \left[\frac{r^3}{3}\right]_{r=0}^{1-\cos\theta} \sin\theta\,d\theta$

$= \int_0^\pi \frac{(1-\cos\theta)^3}{3} \sin\theta\,d\theta = \left[\frac{(1-\cos\theta)^4}{12}\right]_0^\pi = \frac{2^4}{12} = \frac{4}{3}$

6. Compute the line integral $\int \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = (2x, 2y, 2z)$ along the curve $\vec{r}(t) = \left(\frac{2}{t}, \frac{4}{t}, \frac{6}{t}\right)$ from $(2,4,6)$ to $(1,2,3)$.

HINT: Find a scalar potential.

a. -70

b. -42 Correct Choice

c. 0

d. 42

e. 70

Solution: We find a scalar potential.

$\vec{F} = \vec{\nabla} f \qquad \partial_x f = 2x \qquad \partial_y f = 2y \qquad \partial_z f = 2z \quad \Rightarrow \quad f(x,y,z) = x^2 + y^2 + z^2$

By the FTCC, $\int \vec{F} \cdot d\vec{s} = \int \vec{\nabla} f \cdot d\vec{s} = f(1,2,3) - f(2,4,6) = (1^2 + 2^2 + 3^2) - (2^2 + 4^2 + 6^2) = -42$

7. Find the area of the piece of the surface $z = xy$ above the semicircle $x^2 + y^2 \le 9$ for $y \ge 0$. Parametrize the surface as $\vec{R}(u,v) = (u,v,uv)$.
HINT: Find the normal vector.

a. $\frac{\pi}{3}(10^{3/2} - 1)$ Correct Choice

b. $\frac{2\pi}{3}(10^{3/2} - 1)$

c. 9π

d. 18π

e. 36π

Solution: We compute the tangent vectors, the normal vector and its length:

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (1, & 0, & v) \\ \vec{e}_v = & (0, & 1, & u) \end{array} \qquad \vec{N} = \hat{\imath}(0 - v) - \hat{\jmath}(u - 0) + \hat{k}(1) = (-v, -u, 1) \qquad \left|\vec{N}\right| = \sqrt{v^2 + u^2 + 1}$$

$$A = \iint 1\,dS = \int\int \left|\vec{N}\right|\,du\,dv = \int\int \sqrt{v^2 + u^2 + 1}\,du\,dv$$ switch to polar:

$$A = \int_0^\pi \int_0^3 \sqrt{r^2 + 1}\,r\,dr\,d\theta = \pi\left[\frac{(r^2+1)^{3/2}}{3}\right]_0^3 = \frac{\pi}{3}(10^{3/2} - 1)$$

8. Compute $\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy$ for $\vec{F} = (P,Q) = (\sec(x^3) - 5y, \cos(y^5) + 3x)$ counterclockwise around the triangle with vertices $(0,0)$, $(8,0)$ and $(0,4)$.
Hint: Use Green's Theorem.

a. 12
b. 16
c. 32
d. 64
e. 128 Correct Choice

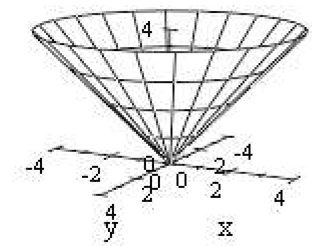
Solution: $P = \sec(x^3) - 5y \qquad Q = \cos(y^5) + 3x \qquad \partial_x Q - \partial_y P = 3 - (-5) = 8$

By Green's Theorem, $\oint \vec{F} \cdot d\vec{s} = \iint \partial_x Q - \partial_y P\,dx\,dy = \iint 8\,dx\,dy = 8\,\text{Area} = 8 \cdot \frac{1}{2} \cdot 8 \cdot 4 = 128$

9. Compute $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the cone $z = \sqrt{x^2 + y^2}$ for $z \le 4$ oriented down and out for $\vec{F} = (y\sqrt{z}, -x\sqrt{z}, \sqrt{z})$.

Note: The cone may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$.

Hint: Use a Theorem.



a. 4

b. 8π

c. 16

d. 32

e. 64π Correct Choice

Solution: By Stokes' Theorem, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$ oriented clockwise.

At the boundary, $z = \sqrt{x^2 + y^2} = 4$. So the boundary is the circle of radius $r = 4$ at height $z = 4$, parametrized by $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 4)$. The vector field is $\vec{F} = (8\sin\theta, -8\cos\theta, 2)$.

The tangent vector is $\vec{v} = (-4\sin\theta, 4\cos\theta, 0)$. Reverse it: $\vec{v} = (4\sin\theta, -4\cos\theta, 0)$.

So $\vec{F} \cdot \vec{v} = 32\sin^2\theta + 32\cos^2\theta = 32$. And the integral is

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, dt = \int_0^{2\pi} 32\, dt = 64\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (20 points) Consider the surface which is the graph of the equation $xy - xz + yz = 11$.

Letter the parts. Box your answers.

a. Find the normal vector to the surface at the point $P = (3,2,1)$.

Solution: Let $f = xy + xz + yz$. Then $\vec{\nabla} f = \langle y - z, x + z, -x + y \rangle$.
Then the normal at $P = (3,2,1)$ is $\vec{N} = \vec{\nabla} f\big|_P = \boxed{\langle 1,4,-1 \rangle}$.

b. Find the standard equation of the tangent plane to the surface at the point $P = (3,2,1)$. Then find its z-intercept.

Solution: The equation of the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $x + 4y - z = 3 + 4 \cdot 2 - 1$ or $\boxed{x + 4y - z = 8}$

Solve for $z = x + 4y - 8$. So the z-intercept is $\boxed{c = -8}$.

c. Find the parametric equation of the normal line to the surface at the point $P = (3,2,1)$. Then find where the normal line intersects the xy-plane.

Solution: The equation of a line is $X = P + t\vec{v}$ where the direction $\vec{v}$ is the normal $\vec{v} = \vec{N} = \langle 1,4,-1 \rangle$.
So $(x,y,z) = (3,2,1) + t\langle 1,4,-1 \rangle$ or $(x,y,z) = (3 + t, 2 + 4t, 1 - t)$.
The line intersects the xy-plane when $z = 1 - t = 0$ or $t = 1$.
So the intersection point is $(x,y,z) = (3 + 1, 2 + 4 \cdot 1, 1 - 1) = \boxed{(4,6,0)}$.

11. (20 points) The temperature around a candle is given by $T = 110 - x^2 - y^2 - 2z^2$.

Find the maximum temperature on the plane $4x + 6y + 8z = 42$ and the point where it occur.

Box your answers.

Solution by Eliminating a Variable:

We solve the constraint for $x = \frac{21}{2} - \frac{3}{2}y - 2z$ and plug into the temperature:

$$T = \left(\frac{21}{2} - \frac{3}{2}y - 2z\right)^2 - y^2 - 2z^2$$

We set the derivatves equal to 0 and solve:

$T_y = -2\left(\frac{21}{2} - \frac{3}{2}y - 2z\right)\left(-\frac{3}{2}\right) - 2y = 0 \qquad T_z = -2\left(\frac{21}{2} - \frac{3}{2}y - 2z\right)(-2) - 4z = 0$

or $\frac{63}{2} - \frac{13}{2}y - 6z = 0 \qquad 42 - 6y - 12z = 0$

or $13y + 12z = 63 \qquad 6y + 12z = 42$

We subtract and substitute back: $7y = 21 \quad y = 3$

$12z = 42 - 6y = 42 - 18 = 24 \qquad z = 2$

$x = \frac{21}{2} - \frac{3}{2}y - 2z = \frac{21}{2} - \frac{9}{2} - 4 = 2$

So: $\boxed{(x,y,z) = (2,3,2)} \quad T = 110 - 4 - 9 - 8 = \boxed{89}$

Solution by Lagrange Multipliers:

We minimize the temperature $T = 110 - x^2 - y^2 - 2z^2$ subject to the constraint $g = 4x + 6y + 8z = 42$.

The gradients are:

$$\vec{\nabla}T = (-2x, -2y, -4z) \qquad \vec{\nabla}g = (4,6,8)$$

So the Lagrange equations are $\nabla T = \lambda \nabla g$ or

$$-2x = 4\lambda \qquad -2y = 6\lambda \qquad -4z = 8\lambda$$

or

$$x = -2\lambda \qquad y = -3\lambda \qquad z = -2\lambda$$

Plug into constraint and solve for λ:

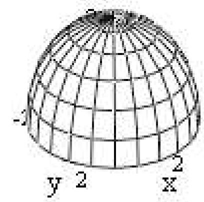
$$4(-2\lambda) + 6(-3\lambda) + 8(-2\lambda) = 42 \qquad -42\lambda = 42 \qquad \lambda = -1$$

So: $\boxed{(x,y,z) = (2,3,2)} \quad T = 110 - 4 - 9 - 8 = \boxed{89}$

12. (20 points) Verify Gauss' Theorem $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$

for the vector field $\vec{F} = \langle xz^2, yz^2, x^2 + y^2 \rangle$ and

the volume inside the hemisphere H: $0 \le z \le \sqrt{4 - x^2 - y^2}$



Be sure to check and explain the orientations. Use the following steps.

Letter the parts. Box your answers.

LHS:

a. Compute the divergence $\vec{\nabla} \cdot \vec{F}$ in rectangular coordinates.

Solution: $\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + 0 = \boxed{2z^2}$

b. What coordinate system will you use to compute the integral $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV$?

What is $\vec{\nabla} \cdot \vec{F}$ in those coordinates?

What is dV in those coordinates?

Solution: Spherical coordinates.

$\vec{\nabla} \cdot \vec{F} = 2z^2 = \boxed{2\rho^2 \cos^2\varphi}$

$dV = \boxed{\rho^2 \sin\varphi \, d\rho d\varphi \, d\theta}$

c. Compute the integral $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV$.

Solution: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 2\rho^2 \cos^2\varphi \, \rho^2 \sin\varphi \, d\rho d\varphi \, d\theta = 2\pi \left[-\frac{\cos^3\varphi}{3} \right]_0^{\pi/2} \left[\frac{2\rho^5}{5} \right]_0^2$

$= 2\pi \left(0 - -\frac{1}{3} \right) \left(\frac{64}{5} \right) = \boxed{\frac{128}{15}\pi}$

RHS:

d. The disk at the bottom, D, may be parametrized as $\vec{R} = (r\cos\theta, r\sin\theta, 0)$.
What is $\vec{F}$ on the disk?

Solution: $\vec{F} = \langle xz^2, yz^2, x^2 + y^2 \rangle = \boxed{\langle 0, 0, r^2 \rangle}$

e. Find the normal to the disk.

Solution:
$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 0) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{array}$$

$\vec{N} = \vec{e}_\theta \times \vec{e}_z = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = \langle 0, 0, r \rangle$

$\vec{N}$ the orientation is up. It should be down. Reverse it: $\vec{N} = \boxed{\langle 0, 0, -r \rangle}$

f. Compute $\iint_D \vec{F}\cdot d\vec{S}$

Solution: $\vec{F}\cdot\vec{N} = -r^3$

$$\iint_D \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{N}\,dr\,d\theta = -\int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = -2\pi\left[\frac{r^4}{4}\right]_0^2 = \boxed{-8\pi}$$

g. The hemisphere, H, may be parametrize as $\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$.
What is $\vec{F}$ on the hemisphere?

Solution: $\vec{F} = \langle xz^2, yz^2, x^2+y^2\rangle = \langle 2\sin\varphi\cos\theta 4\cos^2\varphi, 2\sin\varphi\sin\theta 4\cos^2\varphi, 4\sin^2\varphi\cos^2\theta + 4\sin^2\varphi\sin^2\theta\rangle$
$= \boxed{\langle 8\sin\varphi\cos^2\varphi\cos\theta, 8\sin\varphi\cos^2\varphi\sin\theta, 4\sin^2\varphi\rangle}$

h. Find the normal to the hemisphere.

Solution:
$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\varphi = & (2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi) \\ \vec{e}_\theta = & (-2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0) \end{vmatrix}$$

$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = \hat{\imath}(0 - -4\sin^2\varphi\cos\theta) - \hat{\jmath}(0 - 4\sin^2\varphi\sin\theta) + \hat{k}(4\sin\varphi\cos\varphi\cos^2\theta - -4\sin\varphi\cos\varphi\sin^2\theta)$
$= \boxed{\langle 4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi\rangle}$

$\vec{N}$ the orientation is up as it should be.

i. Compute $\iint_H \vec{F}\cdot d\vec{S}$. HINT: What is $\int_0^{\pi/2}\cos^n\varphi\sin\varphi\,d\varphi$?

Solution: $\vec{F}\cdot\vec{N} = 8\sin\varphi\cos^2\varphi\cos\theta 4\sin^2\varphi\cos\theta + 8\sin\varphi\cos^2\varphi\sin\theta 4\sin^2\varphi\sin\theta + 4\sin^2\varphi 4\sin\varphi\cos\varphi$
$= 32\sin^3\varphi\cos^2\varphi(\cos^2\theta + \sin^2\theta) + 16\sin^3\varphi\cos\varphi$
$= 32\sin\varphi(1-\cos^2\varphi)\cos^2\varphi + 16\sin\varphi(1-\cos^2\varphi)\cos\varphi$
$= 32\sin\varphi(\cos^2\varphi - \cos^4\varphi) + 16\sin\varphi(\cos\varphi - \cos^3\varphi)$
$= 16(2\cos^2\varphi - 2\cos^4\varphi + \cos\varphi - \cos^3\varphi)\sin\varphi$

Notice: $\int_0^{\pi/2}\cos^n\varphi\sin\varphi\,d\varphi = -\int_1^0 u^n\,du = -\left[\frac{u^{n+1}}{n+1}\right]_1^0 = -\frac{0-1}{n+1} = \frac{1}{n+1}$

$$\iint_H \vec{F}\cdot d\vec{S} = \iint_H \vec{F}\cdot\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} 16(2\cos^2\varphi - 2\cos^4\varphi + \cos\varphi - \cos^3\varphi)\sin\varphi\,d\varphi\,d\theta$$
$$= 2\pi 16\left(2\frac{1}{3} - 2\frac{1}{5} + \frac{1}{2} - \frac{1}{4}\right) = 32\pi\frac{40-24+30-15}{60} = 8\pi\frac{31}{15} = \boxed{\frac{248}{15}\pi}$$

j. Combine $\iint_D \vec{F}\cdot d\vec{S}$ and $\iint_H \vec{F}\cdot d\vec{S}$ to get $\iint_{\partial V} \vec{F}\cdot d\vec{S}$.

Solution: $\iint_{\partial V} \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot d\vec{S} + \iint_H \vec{F}\cdot d\vec{S} = -8\pi + 8\pi\frac{31}{15} = 8\pi\left(\frac{-15}{15} + \frac{31}{15}\right) == 8\pi\frac{16}{15} = \boxed{\frac{128}{15}\pi}$

They agree!