MATH 304 Linear Algebra Lecture 9: Determinants.

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ii})_{1 \le i,j \le n}$ is denoted det A or

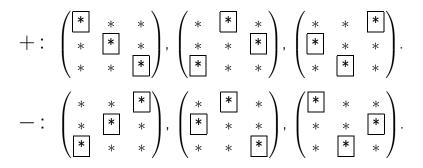
a_{11}	a ₁₂		a _{1n}	
a 21	a 22	•••	a 2n	
÷	÷	•••	÷	•
a _{n1}	a _{n2}	•••	a _{nn}	

Principal property: det A = 0 if and only if the matrix A is singular.

Definition in low dimensions

Definition. det (a) = a,
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
,
 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$

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Examples: 2×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$
$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$
$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$
$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

Examples: 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 - \\ -0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5, \\ \begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 - \\ -6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6. \end{vmatrix}$$

General definition

The general definition of the determinant is quite complicated as there are no simple explicit formula.

There are several approaches to defining determinants. **Approach 1 (original):** an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1) \times (n-1)$ matrices.

 $\mathcal{M}_n(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function det : $\mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

• if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;

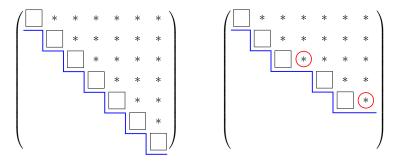
• if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

• if we interchange two rows of a matrix, the determinant changes its sign;

• det I = 1.

Corollary det A = 0 if and only if the matrix A is singular.

Row echelon form of a square matrix:



invertible case

noninvertible case

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det $A = ?$

In the previous lecture we have transformed the matrix A into the identity matrix using elementary row operations.

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -1/2,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -2/5,
- add -3/2 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det $A = ?$

In the previous lecture we have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by -1/2 and by -2/5, and one row exchange.

It follows that

Н

det
$$I = -(-\frac{1}{2})(-\frac{2}{5})$$
 det $A = -\frac{1}{5}$ det A .
ence det $A = -5$ det $I = -5$.

Other properties of determinants

• If a matrix A has two identical rows then det A = 0.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

• If a matrix A has two rows proportional then det A = 0.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

Distributive law for rows

• Suppose that matrices X, Y, Z are identical except for the *i*th row and the *i*th row of Z is the sum of the *i*th rows of X and Y.

Then det $Z = \det X + \det Y$.

$$\begin{vmatrix} a_1 + a_1' & a_2 + a_2' & a_3 + a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

• Adding a scalar multiple of one row to another row does not change the determinant of a matrix.

$$\begin{vmatrix} a_1 + rb_1 & a_2 + rb_2 & a_3 + rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \\ = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} rb_1 & rb_2 & rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Definition. A square matrix $A = (a_{ij})$ is called upper triangular if all entries below the main diagonal are zeros: $a_{ij} = 0$ whenever i > j.

• The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

• If $A = \operatorname{diag}(d_1, d_2, \dots, d_n)$ then det $A = d_1 d_2 \dots d_n$. In particular, det I = 1.

Determinant of the transpose

• If A is a square matrix then det $A^T = \det A$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Columns vs. rows

• If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.

• Interchanging two columns of a matrix changes the sign of its determinant.

• If a matrix A has two columns proportional then $\det A = 0$.

• Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Submatrices

Definition. Given a matrix A, a $k \times k$ submatrix of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

If A is an $n \times n$ matrix then M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the *i*th row and the *j*th column.

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.
 $M_{11} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$, $M_{12} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$, $M_{13} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$,
 $M_{21} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}$, $M_{22} = \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}$, $M_{23} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$,
 $M_{31} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, $M_{32} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$, $M_{33} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$.

Row and column expansions

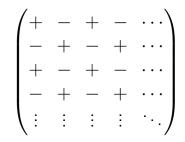
Theorem Let $A = (a_{ij})$ be an $n \times n$ matrix. Then for any $1 \le k, m \le n$ we have that

$$\det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by kth row)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}$$
(expansion by mth column)

Signs for row/column expansions



Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.

Expansion by the 1st row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -5.$$

Expansion by the 2nd row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1 \begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$

Example.
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.

Expansion by the 2nd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -(-2) \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = -5.$$

Expansion by the 3rd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1 \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$