MATH 304

Linear Algebra

Lecture 9: Determinants.

Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix $A = (a_{ij})_{1 \le i,j \le n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is singular.

Definition in low dimensions

Definition. $\det(a) = a$, $\quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \quad \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \quad \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Examples: 2×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$

$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$

$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$

$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

Examples: 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 -$$

$$- 0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5,$$

$$\begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 -$$

$$- 6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6.$$

General definition

The general definition of the determinant is quite complicated as there are no simple explicit formula.

There are several approaches to defining determinants.

Approach 1 (original): an explicit (but very complicated) formula.

Approach 2 (axiomatic): we formulate properties that the determinant should have.

Approach 3 (inductive): the determinant of an $n \times n$ matrix is defined in terms of determinants of certain $(n-1) \times (n-1)$ matrices.

$\mathcal{M}_n(\mathbb{R})$: the set of $n \times n$ matrices with real entries.

Theorem There exists a unique function $\det : \mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$ (called the determinant) with the following properties:

- if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign;
- $\det I = 1$.

Corollary $\det A = 0$ if and only if the matrix A is singular.

Row echelon form of a square matrix:

$$\begin{pmatrix} \Box & * & * & * & * & * & * \\ & \Box & * & * & * & * & * \\ & & \Box & * & * & * & * \\ & & & \Box & * & * & * \\ & & & & \Box & * & * \\ & & & & & \Box & * \\ & & & & & & \Box \end{pmatrix} \qquad \begin{pmatrix} \Box & * & * & * & * & * & * \\ & \Box & * & * & * & * & * \\ & & \Box & * & * & * & * \\ & & & & \Box & * & * \\ & & & & & \Box & * \\ & & & & & & \\ & & & & & & \end{pmatrix}$$

invertible case $\qquad\qquad$ noninvertible case

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

In the previous lecture we have transformed the matrix A into the identity matrix using elementary row operations.

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by $-1/2$,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by $-2/5$,
- add $-3/2$ times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$, $\det A = ?$

In the previous lecture we have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by $-1/2$ and by $-2/5$, and one row exchange.

It follows that

$$\det I = -\left(-\tfrac{1}{2}\right)\left(-\tfrac{2}{5}\right)\det A = -\tfrac{1}{5}\det A.$$

Hence $\det A = -5 \det I = -5.$

Other properties of determinants

- If a matrix A has two identical rows then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

- If a matrix A has two rows proportional then $\det A = 0$.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

Distributive law for rows

- Suppose that matrices X, Y, Z are identical except for the ith row and the ith row of Z is the sum of the ith rows of X and Y.

  Then $\boxed{\det Z = \det X + \det Y.}$

$$\begin{vmatrix} a_1+a_1' & a_2+a_2' & a_3+a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Adding a scalar multiple of one row to another row does not change the determinant of a matrix.

$$\begin{vmatrix} a_1 + rb_1 & a_2 + rb_2 & a_3 + rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} rb_1 & rb_2 & rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Definition. A square matrix $A = (a_{ij})$ is called **upper triangular** if all entries below the main diagonal are zeros: $a_{ij} = 0$ whenever $i > j$.

• The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

• If $A = \mathrm{diag}(d_1, d_2, \dots, d_n)$ then $\det A = d_1 d_2 \dots d_n$. In particular, $\det I = 1$.

Determinant of the transpose

- If A is a square matrix then $\det A^T = \det A$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Columns vs. rows

- If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.
- Interchanging two columns of a matrix changes the sign of its determinant.
- If a matrix A has two columns proportional then $\det A = 0$.
- Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Submatrices

Definition. Given a matrix A, a $k \times k$ **submatrix** of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

If A is an $n\times n$ matrix then M_{ij} denote the $(n-1)\times(n-1)$ submatrix obtained by deleting the ith row and the jth column.

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

$$M_{11} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix},\ M_{12} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix},\ M_{13} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix},\ M_{22} = \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix},\ M_{23} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

$$M_{31} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix},\ M_{32} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix},\ M_{33} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

Row and column expansions

Theorem Let $A = (a_{ij})$ be an $n \times n$ matrix. Then for any $1 \le k, m \le n$ we have that

$$\det A = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det M_{kj},$$

(*expansion by* k*th row*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}.$$

(*expansion by* m*th column*)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

Expansion by the 1st row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -5.$$

Expansion by the 2nd row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1 \begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$.

Expansion by the 2nd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -(-2)\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = -5.$$

Expansion by the 3rd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1\begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$