MATH 304 Linear Algebra Lecture 9: Determinants.

### Determinants

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix  $A = (a_{ii})_{1 \le i,j \le n}$  is denoted det A or

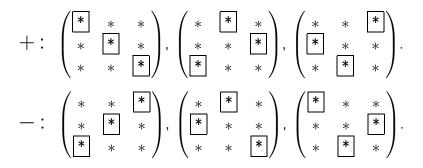
$a_{11}$	<b>a</b> <sub>12</sub>		a <sub>1n</sub>	
<b>a</b> 21	<b>a</b> 22	•••	<b>a</b> 2n	
÷	÷	•••	÷	•
a <sub>n1</sub>	a <sub>n2</sub>	•••	a <sub>nn</sub>	

**Principal property:** det A = 0 if and only if the matrix A is singular.

# Definition in low dimensions

Definition. det (a) = a, 
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
,  
 $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$ 

÷.



# **Examples: 2**×2 matrices

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \qquad \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = -12,$$
$$\begin{vmatrix} -2 & 5 \\ 0 & 3 \end{vmatrix} = -6, \qquad \begin{vmatrix} 7 & 0 \\ 5 & 2 \end{vmatrix} = 14,$$
$$\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1, \qquad \begin{vmatrix} 0 & 0 \\ 4 & 1 \end{vmatrix} = 0,$$
$$\begin{vmatrix} -1 & 3 \\ -1 & 3 \end{vmatrix} = 0, \qquad \begin{vmatrix} 2 & 1 \\ 8 & 4 \end{vmatrix} = 0.$$

# **Examples:** 3×3 matrices

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \cdot 0 \cdot 0 + (-2) \cdot 1 \cdot (-2) + 0 \cdot 1 \cdot 3 - \\ -0 \cdot 0 \cdot (-2) - (-2) \cdot 1 \cdot 0 - 3 \cdot 1 \cdot 3 = 4 - 9 = -5, \\ \begin{vmatrix} 1 & 4 & 6 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 + 4 \cdot 5 \cdot 0 + 6 \cdot 0 \cdot 0 - \\ -6 \cdot 2 \cdot 0 - 4 \cdot 0 \cdot 3 - 1 \cdot 5 \cdot 0 = 1 \cdot 2 \cdot 3 = 6. \end{vmatrix}$$

# **General definition**

The general definition of the determinant is quite complicated as there are no simple explicit formula.

There are several approaches to defining determinants. **Approach 1 (original):** an explicit (but very complicated) formula.

**Approach 2 (axiomatic):** we formulate properties that the determinant should have.

**Approach 3 (inductive):** the determinant of an  $n \times n$  matrix is defined in terms of determinants of certain  $(n-1) \times (n-1)$  matrices.

 $\mathcal{M}_n(\mathbb{R})$ : the set of  $n \times n$  matrices with real entries.

**Theorem** There exists a unique function det :  $\mathcal{M}_n(\mathbb{R}) \to \mathbb{R}$  (called the determinant) with the following properties:

• if a row of a matrix is multiplied by a scalar r, the determinant is also multiplied by r;

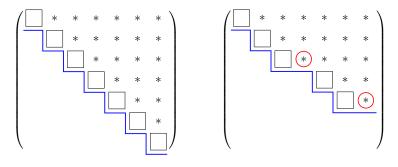
• if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;

• if we interchange two rows of a matrix, the determinant changes its sign;

• det I = 1.

**Corollary** det A = 0 if and only if the matrix A is singular.

Row echelon form of a square matrix:



invertible case

noninvertible case

Example. 
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det  $A = ?$ 

In the previous lecture we have transformed the matrix A into the identity matrix using elementary row operations.

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -1/2,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -2/5,
- add -3/2 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

Example. 
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
, det  $A = ?$ 

In the previous lecture we have transformed the matrix A into the identity matrix using elementary row operations.

These included two row multiplications, by -1/2 and by -2/5, and one row exchange.

It follows that

Н

det 
$$I = -(-\frac{1}{2})(-\frac{2}{5})$$
 det  $A = -\frac{1}{5}$  det  $A$ .  
ence det  $A = -5$  det  $I = -5$ .

### Other properties of determinants

• If a matrix A has two identical rows then det A = 0.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

• If a matrix A has two rows proportional then det A = 0.

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ ra_1 & ra_2 & ra_3 \end{vmatrix} = r \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = 0$$

#### Distributive law for rows

• Suppose that matrices X, Y, Z are identical except for the *i*th row and the *i*th row of Z is the sum of the *i*th rows of X and Y.

Then det  $Z = \det X + \det Y$ .

$$\begin{vmatrix} a_1 + a_1' & a_2 + a_2' & a_3 + a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & a_2' & a_3' \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

• Adding a scalar multiple of one row to another row does not change the determinant of a matrix.

$$\begin{vmatrix} a_1 + rb_1 & a_2 + rb_2 & a_3 + rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \\ = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} rb_1 & rb_2 & rb_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Definition. A square matrix  $A = (a_{ij})$  is called upper triangular if all entries below the main diagonal are zeros:  $a_{ij} = 0$  whenever i > j.

• The determinant of an upper triangular matrix is equal to the product of its diagonal entries.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33}$$

• If  $A = \operatorname{diag}(d_1, d_2, \dots, d_n)$  then det  $A = d_1 d_2 \dots d_n$ . In particular, det I = 1.

#### **Determinant of the transpose**

• If A is a square matrix then det  $A^T = \det A$ .

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

#### Columns vs. rows

• If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.

• Interchanging two columns of a matrix changes the sign of its determinant.

• If a matrix A has two columns proportional then  $\det A = 0$ .

• Adding a scalar multiple of one column to another does not change the determinant of a matrix.

#### **Submatrices**

Definition. Given a matrix A, a  $k \times k$  submatrix of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

If A is an  $n \times n$  matrix then  $M_{ij}$  denote the  $(n-1) \times (n-1)$  submatrix obtained by deleting the *i*th row and the *j*th column.

Example. 
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.  
 $M_{11} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$ ,  $M_{12} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$ ,  $M_{13} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$ ,  
 $M_{21} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}$ ,  $M_{22} = \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}$ ,  $M_{23} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ ,  
 $M_{31} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $M_{32} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ ,  $M_{33} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}$ .

#### Row and column expansions

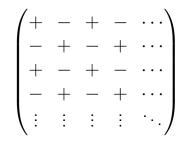
**Theorem** Let  $A = (a_{ij})$  be an  $n \times n$  matrix. Then for any  $1 \le k, m \le n$  we have that

$$\det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by kth row)

$$\det A = \sum_{i=1}^{n} (-1)^{i+m} a_{im} \det M_{im}$$
(expansion by mth column)

# Signs for row/column expansions



Example. 
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.

Expansion by the 1st row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = 3 \begin{vmatrix} 0 & 1 \\ 3 & 0 \end{vmatrix} - (-2) \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = -5.$$

Expansion by the 2nd row:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1 \begin{vmatrix} -2 & 0 \\ 3 & 0 \end{vmatrix} - 1 \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$

Example. 
$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}$$
.

Expansion by the 2nd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -(-2) \begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} - 3 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = -5.$$

Expansion by the 3rd column:

$$\begin{vmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{vmatrix} = -1 \begin{vmatrix} 3 & -2 \\ -2 & 3 \end{vmatrix} = -5.$$