MATH 304

Linear Algebra

# Lecture 18:
# Rank and nullity of a matrix.

# Nullspace

Let $A = (a_{ij})$ be an $m \times n$ matrix.

*Definition.* The **nullspace** of the matrix $A$, denoted $N(A)$, is the set of all $n$-dimensional column vectors $\mathbf{x}$ such that $\boxed{A\mathbf{x} = \mathbf{0}.}$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The nullspace $N(A)$ is the solution set of a system of linear homogeneous equations (with $A$ as the coefficient matrix).

Let $A = (a_{ij})$ be an $m \times n$ matrix.

**Theorem** The nullspace $N(A)$ is a subspace of the vector space $\mathbb{R}^n$.

*Proof:* We have to show that $N(A)$ is nonempty, closed under addition, and closed under scaling.

First of all, $A\mathbf{0} = \mathbf{0} \implies \mathbf{0} \in N(A) \implies N(A)$ is not empty.

Secondly, if $\mathbf{x}, \mathbf{y} \in N(A)$, i.e., if $A\mathbf{x} = A\mathbf{y} = \mathbf{0}$, then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0} \implies \mathbf{x}+\mathbf{y} \in N(A)$.

Thirdly, if $\mathbf{x} \in N(A)$, i.e., if $A\mathbf{x} = \mathbf{0}$, then for any $r \in \mathbb{R}$ one has $A(r\mathbf{x}) = r(A\mathbf{x}) = r\mathbf{0} = \mathbf{0} \implies r\mathbf{x} \in N(A)$.

*Definition.* The dimension of the nullspace $N(A)$ is called the **nullity** of the matrix $A$.

**Problem.** Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

Elementary row operations do not change the nullspace.
Let us convert $A$ to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 \\ x_2 = -2x_3 - 3x_4 \end{cases}$$

General element of $N(A)$:

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (t + 2s, -2t - 3s, t, s) \\ &= t(1, -2, 1, 0) + s(2, -3, 0, 1), \quad t, s \in \mathbb{R}. \end{aligned}$$

Vectors $(1, -2, 1, 0)$ and $(2, -3, 0, 1)$ form a basis for $N(A)$.
Thus the nullity of the matrix $A$ is 2.

# Row space

*Definition.* The **row space** of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^n$ spanned by rows of $A$.

The dimension of the row space is called the **rank** of the matrix $A$.

**Theorem 1** Elementary row operations do not change the row space of a matrix.

**Theorem 2** If a matrix $A$ is in row echelon form, then the nonzero rows of $A$ are linearly independent.

**Corollary** The rank of a matrix is equal to the number of nonzero rows in its row echelon form.

**Theorem 3** The rank of a matrix $A$ plus the nullity of $A$ equals the number of columns of $A$.

**Problem.** Find the rank of the matrix

$$A = \begin{pmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}.$$

Elementary row operations do not change the row space. Let us convert $A$ to row echelon form:

$$\begin{pmatrix} -1 & 0 & -1 & 2 \\ 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 0 & 1 & -1 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} -1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Vectors $(1, 0, 1, -2)$ and $(0, 0, 0, 1)$ form a basis for the row space of $A$. Thus the rank of $A$ is 2.

*Remark.* The rank of $A$ equals the number of nonzero rows in the row echelon form, which equals the number of leading entries.

The nullity of $A$ equals the number of free variables in the corresponding system, which equals the number of columns without leading entries.

Consequently, rank+nullity is the number of all columns in the matrix $A$.

**Theorem 1** Elementary row operations do not change the row space of a matrix.

*Proof:* Suppose that $A$ and $B$ are $m \times n$ matrices such that $B$ is obtained from $A$ by an elementary row operation. Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be the rows of $A$ and $\mathbf{b}_1, \ldots, \mathbf{b}_m$ be the rows of $B$. We have to show that $\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) = \mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m)$.

Observe that any row $\mathbf{b}_i$ of $B$ belongs to $\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$. Indeed, either $\mathbf{b}_i = \mathbf{a}_j$ for some $1 \leq j \leq m$, or $\mathbf{b}_i = r\mathbf{a}_i$ for some scalar $r \neq 0$, or $\mathbf{b}_i = \mathbf{a}_i + r\mathbf{a}_j$ for some $j \neq i$ and $r \in \mathbb{R}$.

It follows that $\mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m) \subset \mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m)$.

Now the matrix $A$ can also be obtained from $B$ by an elementary row operation. By the above,

$$\mathrm{Span}(\mathbf{a}_1, \ldots, \mathbf{a}_m) \subset \mathrm{Span}(\mathbf{b}_1, \ldots, \mathbf{b}_m).$$

**Problem.** Find the nullity of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}.$$

*Alternative solution:* Clearly, the rows of $A$ are linearly independent. Therefore the rank of $A$ is 2. Since

$$(\text{rank of } A) + (\text{nullity of } A) = 4,$$

it follows that the nullity of $A$ is 2.

# Column space

*Definition.* The **column space** of an $m \times n$ matrix $A$ is the subspace of $\mathbb{R}^m$ spanned by columns of $A$.

**Theorem 1** The column space of a matrix $A$ coincides with the row space of the transpose matrix $A^T$.

**Theorem 2** Elementary column operations do not change the column space of a matrix.

**Theorem 3** Elementary row operations do not change the dimension of the column space of a matrix (although they can change the column space).

**Theorem 4** For any matrix, the row space and the column space have the same dimension.

**Problem.** Find a basis for the column space of the matrix
$$B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{pmatrix}.$$

The column space of $B$ coincides with the row space of $B^T$. To find a basis, we convert $B^T$ to row echelon form:
$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \\ 2 & 3 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 2 & -1 \\ 0 & \color{red}{1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus vectors $(1, 2, -1)$ and $(0, 1, 0)$ form a basis for the column space of the matrix $B$.

**Problem.** Find a basis for the column space of the matrix

$$B = \begin{pmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 2 & 3 \\ -1 & 0 & 1 & -2 \end{pmatrix}.$$

*Alternative solution:* The dimension of the column space equals the dimension of the row space, which is 2 (since the first two rows are not parallel and the third row is a multiple of the first one).

The 1st and the 2nd columns, $(1, 2, -1)$ and $(0, 1, 0)$, are linearly independent. It follows that they form a basis for the column space (actually, any two columns form such a basis).