MATH 304 Linear Algebra

Lecture 23: Similarity of matrices.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

 $\mathbf{v}=x_1\mathbf{v}_1+x_2\mathbf{v}_2+\cdots+x_n\mathbf{v}_n,$

where $x_i \in \mathbb{R}$. The coefficients x_1, x_2, \ldots, x_n are called the **coordinates** of **v** with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

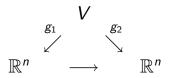
The mapping

vector $\mathbf{v} \mapsto its$ coordinates (x_1, x_2, \ldots, x_n)

provides a one-to-one correspondence between V and \mathbb{R}^n . This mapping is linear.

Change of coordinates

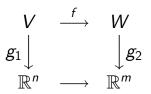
Let V be a vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.



The composition $g_2 \circ g_1^{-1}$ is a linear mapping of \mathbb{R}^n to itself. It is represented as $\mathbf{v} \mapsto U\mathbf{v}$, where U is an $n \times n$ matrix. U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Matrix of a linear mapping

Let V, W be vector spaces and $f: V \to W$ be a linear map. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1: V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a basis for W and $g_2: W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

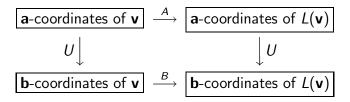


The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of \mathbb{R}^n to \mathbb{R}^m . It is represented as $\mathbf{v} \mapsto A\mathbf{v}$, where A is an $m \times n$ matrix. A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

Change of basis for a linear operator

Let $L: V \to V$ be a linear operator on a vector space V. Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$.



It follows that UA = BU. Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Problem. Let
$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
.

Find the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = A\mathbf{x}$ with respect to the basis $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (-1, 0, 1)$.

Let *B* be the desired matrix. The columns of *B* are coordinates of the vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$L(\mathbf{v}_{1}) = (0, 0, 0), \quad L(\mathbf{v}_{2}) = (2, 2, 0) = 2\mathbf{v}_{2}$$

$$L(\mathbf{v}_{3}) = (-2, 0, 2) = 2\mathbf{v}_{3}.$$

Thus $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$

Problem. Let
$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$
. Find A^{16} .

It follows from the solution of the previous problem that $A = UBU^{-1}$, where

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad U = \begin{pmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that $A^2 = AA = UBU^{-1}UBU^{-1} = UB^2U^{-1}$, $A^3 = A^2A = UB^2U^{-1}UBU^{-1} = UB^3U^{-1}$, and so on. In particular, $A^{16} = UB^{16}U^{-1}$. Clearly, $B^{16} = \text{diag}(0, 2^{16}, 2^{16}) = 2^{15}B$. Hence $A^{16} = U(2^{15}B)U^{-1} = 2^{15}UBU^{-1} = 2^{15}A$ = 32768 A.

$$A^{16} = \begin{pmatrix} 32768 & 32768 & -32768 \\ 32768 & 32768 & 32768 \\ 0 & 0 & 65536 \end{pmatrix}$$

Similarity

Definition. An $n \times n$ matrix B is said to be similar to an $n \times n$ matrix A if $B = S^{-1}AS$ for some nonsingular $n \times n$ matrix S.

Remark. Two $n \times n$ matrices are similar if and only if they represent the same linear operator on \mathbb{R}^n with respect to some bases.

Theorem Similarity is an equivalence relation, i.e.,
(i) any square matrix A is similar to itself;
(ii) if B is similar to A, then A is similar to B;
(iii) if A is similar to B and B is similar to C, then A is similar to C.

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Proof: (i)
$$A = I^{-1}AI$$
.
(ii) If $B = S^{-1}AS$ then $A = SBS^{-1} = (S^{-1})^{-1}BS^{-1}$
(iii) If $A = S^{-1}BS$ and $B = T^{-1}CT$ then
 $A = S^{-1}T^{-1}CTS = (TS)^{-1}C(TS)$.

Theorem If A and B are similar matrices then they have the same (i) determinant, (ii) trace = the sum of diagonal entries, (iii) rank, and (iv) nullity.