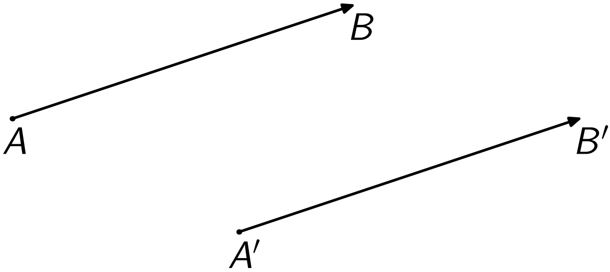
MATH 304

Linear Algebra

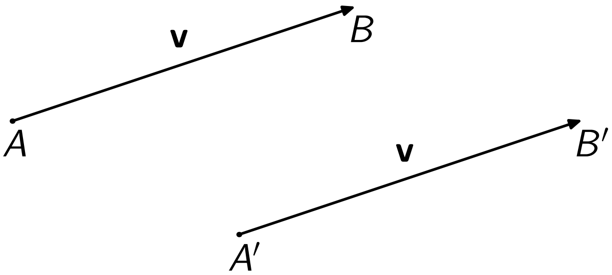
Lecture 24:
Scalar product.

Vectors: geometric approach



- A vector is represented by a directed segment.
- Directed segment is drawn as an arrow.
- Different arrows represent the same vector if they are of the same length and direction.

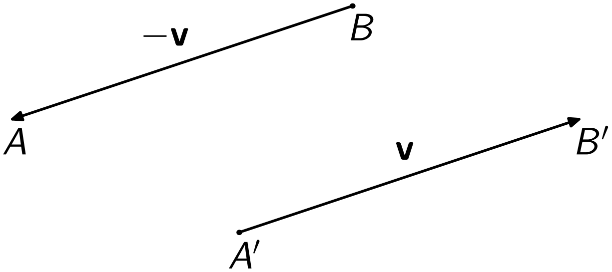
Vectors: geometric approach



$\overrightarrow{AB}$ denotes the vector represented by the arrow with tip at B and tail at A.

$\overrightarrow{AA}$ is called the *zero vector* and denoted $\mathbf{0}$.

Vectors: geometric approach

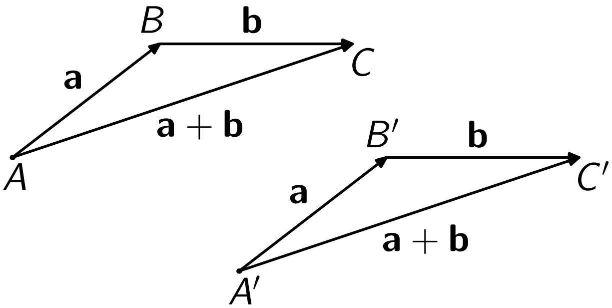


If $\mathbf{v} = \overrightarrow{AB}$ then $\overrightarrow{BA}$ is called the *negative vector* of $\mathbf{v}$ and denoted $-\mathbf{v}$.

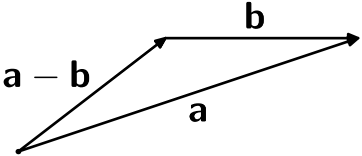
Vector addition

Given vectors $\mathbf{a}$ and $\mathbf{b}$, their *sum* $\mathbf{a} + \mathbf{b}$ is defined by the rule $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

That is, choose points A, B, C so that $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{BC} = \mathbf{b}$. Then $\mathbf{a} + \mathbf{b} = \overrightarrow{AC}$.



The *difference* of the two vectors is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$.



Properties of vector addition:

$\mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a}$ (commutative law)

$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c})$ (associative law)

$\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{a}=\mathbf{a}$

$\mathbf{a}+(-\mathbf{a})=(-\mathbf{a})+\mathbf{a}=\mathbf{0}$

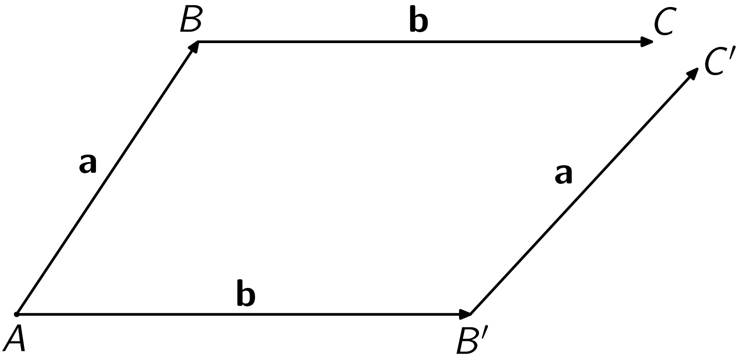
Let $\overrightarrow{AB}=\mathbf{a}$. Then $\mathbf{a}+\mathbf{0}=\overrightarrow{AB}+\overrightarrow{BB}=\overrightarrow{AB}=\mathbf{a}$,
$\mathbf{a}+(-\mathbf{a})=\overrightarrow{AB}+\overrightarrow{BA}=\overrightarrow{AA}=\mathbf{0}$.

Let $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{b}$, and $\overrightarrow{CD}=\mathbf{c}$. Then
$(\mathbf{a}+\mathbf{b})+\mathbf{c}=(\overrightarrow{AB}+\overrightarrow{BC})+\overrightarrow{CD}=\overrightarrow{AC}+\overrightarrow{CD}=\overrightarrow{AD}$,
$\mathbf{a}+(\mathbf{b}+\mathbf{c})=\overrightarrow{AB}+(\overrightarrow{BC}+\overrightarrow{CD})=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$.

Parallelogram rule

Let $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$, $\overrightarrow{AB'} = \mathbf{b}$, and $\overrightarrow{B'C'} = \mathbf{a}$.

Then $\mathbf{a} + \mathbf{b} = \overrightarrow{AC}$, $\mathbf{b} + \mathbf{a} = \overrightarrow{AC'}$.



Wrong picture!

Parallelogram rule

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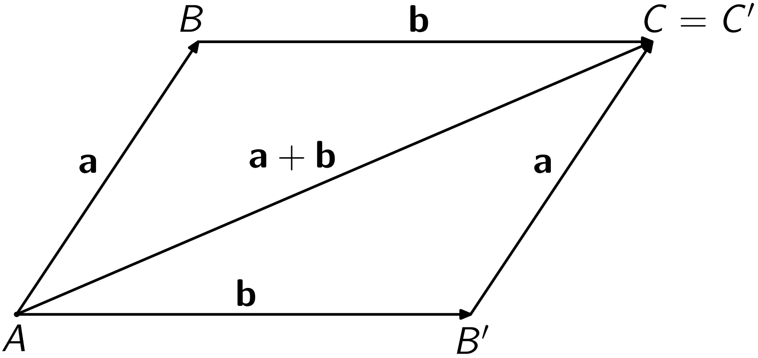
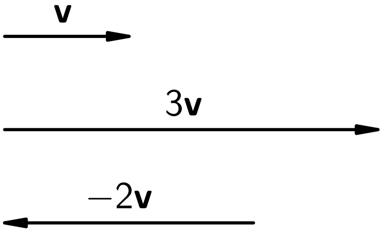


Right picture!

Scalar multiplication

Let $\mathbf{v}$ be a vector and $r \in \mathbb{R}$. By definition, $r\mathbf{v}$ is a vector whose magnitude is $|r|$ times the magnitude of $\mathbf{v}$. The direction of $r\mathbf{v}$ coincides with that of $\mathbf{v}$ if $r > 0$. If $r < 0$ then the directions of $r\mathbf{v}$ and $\mathbf{v}$ are opposite.



Scalar multiplication

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Properties of scalar multiplication:

$r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$ (distributive law #1)

$(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$ (distributive law #2)

$r(s\mathbf{a}) = (rs)\mathbf{a}$ (associative law)

$1\mathbf{a} = \mathbf{a}$

Beyond linearity: length of a vector

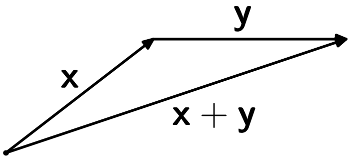
The **length** (or the **magnitude**) of a vector $\overrightarrow{AB}$ is the length of the representing segment AB. The length of a vector $\mathbf{v}$ is denoted $|\mathbf{v}|$ or $\|\mathbf{v}\|$.

Properties of vector length:

$|\mathbf{x}| \geq 0,\ \ |\mathbf{x}| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$|r\mathbf{x}| = |r|\,|\mathbf{x}|$ (homogeneity)

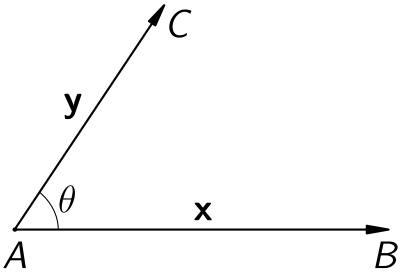
$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ (triangle inequality)

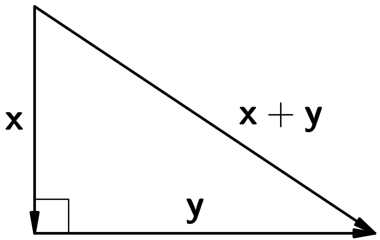


Beyond linearity: angle between vectors

Given nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, let A, B, and C be points such that $\overrightarrow{AB} = \mathbf{x}$ and $\overrightarrow{AC} = \mathbf{y}$. Then $\angle BAC$ is called the **angle** between $\mathbf{x}$ and $\mathbf{y}$.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are called **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if the angle between them equals 90^{o}.





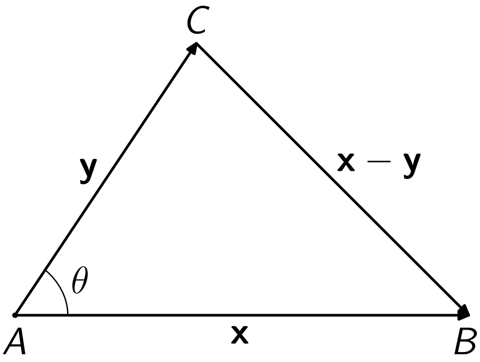
Pythagorean Theorem:

$$\mathbf{x} \perp \mathbf{y} \implies |\mathbf{x}+\mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2$$

3-*dimensional Pythagorean Theorem:*

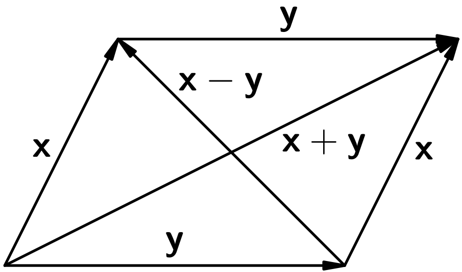
If vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are pairwise orthogonal then

$$|\mathbf{x}+\mathbf{y}+\mathbf{z}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 + |\mathbf{z}|^2$$



Law of cosines:

$$|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2|\mathbf{x}|\,|\mathbf{y}|\,\cos\theta$$



Parallelogram Identity:

$$|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

Beyond linearity: dot product

The **dot product** of vectors $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\,\cos\theta,$$

where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

The dot product is also called the **scalar product**.
Alternative notation: $(\mathbf{x}, \mathbf{y})$ or $\langle \mathbf{x}, \mathbf{y} \rangle$.

The vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Relations between lengths and dot products:

- $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
- $|\mathbf{x} \cdot \mathbf{y}| \le |\mathbf{x}|\,|\mathbf{y}|$
- $|\mathbf{x} - \mathbf{y}|^2 = |\mathbf{x}|^2 + |\mathbf{y}|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$

Vectors: algebraic approach

An n-dimensional coordinate vector is an element of $\mathbb{R}^n$, i.e., an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers.

Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be vectors, and $r \in \mathbb{R}$ be a scalar. Then, by definition,

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),$$

$$r\mathbf{a} = (ra_1, ra_2, \ldots, ra_n),$$

$$\mathbf{0} = (0, 0, \ldots, 0),$$

$$-\mathbf{b} = (-b_1, -b_2, \ldots, -b_n),$$

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n).$$

Properties of vector addition and scalar multiplication:

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$$

$$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$$

$$\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$$

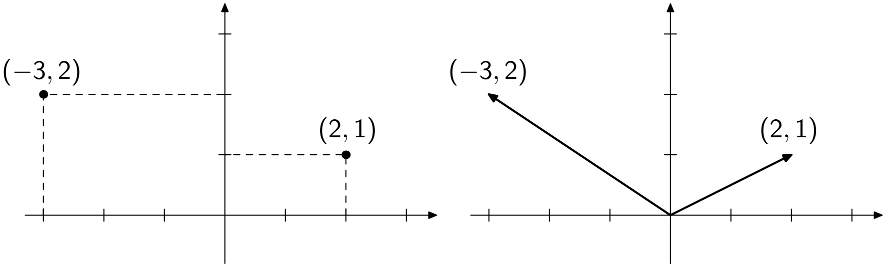
$$r(\mathbf{a} + \mathbf{b}) = r\mathbf{a} + r\mathbf{b}$$

$$(r + s)\mathbf{a} = r\mathbf{a} + s\mathbf{a}$$

$$r(s\mathbf{a}) = (rs)\mathbf{a}$$

$$1\mathbf{a} = \mathbf{a}$$

Cartesian coordinates: geometric meets algebraic



Once we specify an *origin* O, each point A is associated a *position vector* $\overrightarrow{OA}$. Conversely, every vector has a unique representative with tail at O.

Cartesian coordinates allow us to identify a line, a plane, and space with $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$, respectively.

Length and distance

Definition. The **length** of a vector $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

The **distance** between vectors/points $\mathbf{x}$ and $\mathbf{y}$ is $\|\mathbf{y} - \mathbf{x}\|$.

Properties of length:

$\|\mathbf{x}\| \geq 0$, $\|\mathbf{x}\| = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\|r\mathbf{x}\| = |r|\,\|\mathbf{x}\|$ (homogeneity)

$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (triangle inequality)

Scalar product

Definition. The **scalar product** of vectors $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ is

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{k=1}^{n} x_k y_k.$$

Properties of scalar product:

$\mathbf{x} \cdot \mathbf{x} \geq 0$, $\mathbf{x} \cdot \mathbf{x} = 0$ only if $\mathbf{x} = \mathbf{0}$ (positivity)

$\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)

$(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

Relations between lengths and scalar products:

$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \, \|\mathbf{y}\|$ (Cauchy-Schwarz inequality)

$\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x} \cdot \mathbf{y}$

By the Cauchy-Schwarz inequality, for any nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|} \quad \text{for some} \quad 0 \leq \theta \leq \pi.$$

θ is called the **angle** between the vectors **x** and **y**. The vectors **x** and **y** are said to be **orthogonal** (denoted $\mathbf{x} \perp \mathbf{y}$) if $\mathbf{x} \cdot \mathbf{y} = 0$ (i.e., if $\theta = 90^{\circ}$).

Problem. Find the angle θ between vectors $\mathbf{x} = (2, -1)$ and $\mathbf{y} = (3, 1)$.

$\mathbf{x} \cdot \mathbf{y} = 5, \ \ \|\mathbf{x}\| = \sqrt{5}, \ \ \|\mathbf{y}\| = \sqrt{10}.$

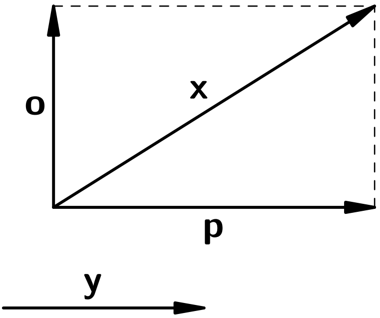
$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\|\,\|\mathbf{y}\|} = \frac{5}{\sqrt{5}\,\sqrt{10}} = \frac{1}{\sqrt{2}} \implies \theta = 45^{\mathrm{o}}$$

Problem. Find the angle ϕ between vectors $\mathbf{v} = (-2, 1, 3)$ and $\mathbf{w} = (4, 5, 1)$.

$$\mathbf{v} \cdot \mathbf{w} = 0 \implies \mathbf{v} \perp \mathbf{w} \implies \phi = 90^{\mathrm{o}}$$

Orthogonal projection

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.



$\mathbf{p}$ = orthogonal projection of $\mathbf{x}$ onto $\mathbf{y}$

Orthogonal projection

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.

Namely, $\mathbf{p} = \alpha\mathbf{u}$, where $\mathbf{u}$ is the unit vector of the same direction as $\mathbf{y}$, and $\alpha = \mathbf{x} \cdot \mathbf{u}$.

Indeed, $\mathbf{p} \cdot \mathbf{u} = (\alpha\mathbf{u}) \cdot \mathbf{u} = \alpha(\mathbf{u} \cdot \mathbf{u}) = \alpha\|\mathbf{u}\|^2 = \alpha = \mathbf{x} \cdot \mathbf{u}$.
Hence $\mathbf{o} \cdot \mathbf{u} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} - \mathbf{p} \cdot \mathbf{u} = 0 \implies \mathbf{o} \perp \mathbf{u}$
$\implies \mathbf{o} \perp \mathbf{y}$.

$\mathbf{p}$ is called the **vector projection** of $\mathbf{x}$ onto $\mathbf{y}$,
$\alpha = \pm\|\mathbf{p}\|$ is called the **scalar projection** of $\mathbf{x}$ onto $\mathbf{y}$.

$$\mathbf{u} = \frac{\mathbf{y}}{\|\mathbf{y}\|}, \qquad \alpha = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|}, \qquad \mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\mathbf{y}.$$

Problem. Find the distance from the point $\mathbf{x} = (3, 1)$ to the line spanned by $\mathbf{y} = (2, -1)$.

Consider the decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p}$ is parallel to $\mathbf{y}$ while $\mathbf{o} \perp \mathbf{y}$. The required distance is the length of the orthogonal component $\mathbf{o}$.

$$\mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\, \mathbf{y} = \frac{5}{5}\,(2, -1) = (2, -1),$$

$$\mathbf{o} = \mathbf{x} - \mathbf{p} = (3, 1) - (2, -1) = (1, 2), \qquad \|\mathbf{o}\| = \sqrt{5}.$$

Problem. Find the point on the line $y = -x$ that is closest to the point $(3, 4)$.

The required point is the projection $\mathbf{p}$ of $\mathbf{v} = (3, 4)$ on the vector $\mathbf{w} = (1, -1)$ spanning the line $y = -x$.

$$\mathbf{p} = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\, \mathbf{w} = \frac{-1}{2}\,(1, -1) = \left(-\frac{1}{2}, \frac{1}{2}\right)$$