

MATH 304

Linear Algebra

Lecture 29:

Orthogonal sets.

The Gram-Schmidt process.

Orthogonal sets

Let V be an inner product space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$.

Definition. A nonempty set $S \subset V$ of nonzero vectors is called an **orthogonal set** if all vectors in S are mutually orthogonal. That is, $\mathbf{0} \notin S$ and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ for any $\mathbf{x}, \mathbf{y} \in S$, $\mathbf{x} \neq \mathbf{y}$.

An orthogonal set $S \subset V$ is called **orthonormal** if $\|\mathbf{x}\| = 1$ for any $\mathbf{x} \in S$.

Remark. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ form an orthonormal set if and only if

$$\langle \mathbf{v}_i, \mathbf{v}_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Examples. • $V = \mathbb{R}^n$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

The standard basis $\mathbf{e}_1 = (1, 0, 0, \dots, 0)$,
 $\mathbf{e}_2 = (0, 1, 0, \dots, 0)$, \dots , $\mathbf{e}_n = (0, 0, 0, \dots, 1)$.

It is an orthonormal set.

• $V = \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y}$.

$\mathbf{v}_1 = (3, 5, 4)$, $\mathbf{v}_2 = (3, -5, 4)$, $\mathbf{v}_3 = (4, 0, -3)$.

$\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, $\mathbf{v}_1 \cdot \mathbf{v}_3 = 0$, $\mathbf{v}_2 \cdot \mathbf{v}_3 = 0$,

$\mathbf{v}_1 \cdot \mathbf{v}_1 = 50$, $\mathbf{v}_2 \cdot \mathbf{v}_2 = 50$, $\mathbf{v}_3 \cdot \mathbf{v}_3 = 25$.

Thus the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is orthogonal but not orthonormal. An orthonormal set is formed by

normalized vectors $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$,

$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$.

- $V = C[-\pi, \pi]$, $\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) dx$.

$$f_1(x) = \sin x, f_2(x) = \sin 2x, \dots, f_n(x) = \sin nx, \dots$$

$$\begin{aligned}\langle f_m, f_n \rangle &= \int_{-\pi}^{\pi} \sin(mx) \sin(nx) dx \\ &= \int_{-\pi}^{\pi} \frac{1}{2} (\cos(mx - nx) - \cos(mx + nx)) dx.\end{aligned}$$

$$\int_{-\pi}^{\pi} \cos(kx) dx = \frac{\sin(kx)}{k} \Big|_{x=-\pi}^{\pi} = 0 \text{ if } k \in \mathbb{Z}, k \neq 0.$$

$$k = 0 \implies \int_{-\pi}^{\pi} \cos(kx) dx = \int_{-\pi}^{\pi} dx = 2\pi.$$

$$\begin{aligned}\langle f_m, f_n \rangle &= \frac{1}{2} \int_{-\pi}^{\pi} (\cos(m-n)x - \cos(m+n)x) dx \\ &= \begin{cases} \pi & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}\end{aligned}$$

Thus the set $\{f_1, f_2, f_3, \dots\}$ is orthogonal but not orthonormal.

It is orthonormal with respect to a scaled inner product

$$\langle\langle f, g \rangle\rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x) dx.$$

Orthogonality \implies linear independence

Theorem Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are nonzero vectors that form an orthogonal set. Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly independent.

Proof: Suppose $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_k\mathbf{v}_k = \mathbf{0}$ for some $t_1, t_2, \dots, t_k \in \mathbb{R}$.

Then for any index $1 \leq i \leq k$ we have

$$\langle t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_k\mathbf{v}_k, \mathbf{v}_i \rangle = \langle \mathbf{0}, \mathbf{v}_i \rangle = 0.$$

$$\implies t_1\langle \mathbf{v}_1, \mathbf{v}_i \rangle + t_2\langle \mathbf{v}_2, \mathbf{v}_i \rangle + \dots + t_k\langle \mathbf{v}_k, \mathbf{v}_i \rangle = 0$$

By orthogonality, $t_i\langle \mathbf{v}_i, \mathbf{v}_i \rangle = 0 \implies t_i = 0$.

Orthonormal bases

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be an orthonormal basis for an inner product space V .

Theorem Let $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \dots + x_n\mathbf{v}_n$ and $\mathbf{y} = y_1\mathbf{v}_1 + y_2\mathbf{v}_2 + \dots + y_n\mathbf{v}_n$, where $x_i, y_j \in \mathbb{R}$. Then

(i) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n$,

(ii) $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$.

Proof: (ii) follows from (i) when $\mathbf{y} = \mathbf{x}$.

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle &= \left\langle \sum_{i=1}^n x_i \mathbf{v}_i, \sum_{j=1}^n y_j \mathbf{v}_j \right\rangle = \sum_{i=1}^n x_i \left\langle \mathbf{v}_i, \sum_{j=1}^n y_j \mathbf{v}_j \right\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n x_i y_j \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \sum_{i=1}^n x_i y_i.\end{aligned}$$

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for an inner product space V .

Theorem If the basis $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal set then for any $\mathbf{x} \in V$

$$\mathbf{x} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle} \mathbf{v}_n.$$

If $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthonormal set then

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{x}, \mathbf{v}_2 \rangle \mathbf{v}_2 + \cdots + \langle \mathbf{x}, \mathbf{v}_n \rangle \mathbf{v}_n.$$

Proof: We have that $\mathbf{x} = x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n$.

$$\implies \langle \mathbf{x}, \mathbf{v}_i \rangle = \langle x_1 \mathbf{v}_1 + \cdots + x_n \mathbf{v}_n, \mathbf{v}_i \rangle, \quad 1 \leq i \leq n.$$

$$\implies \langle \mathbf{x}, \mathbf{v}_i \rangle = x_1 \langle \mathbf{v}_1, \mathbf{v}_i \rangle + \cdots + x_n \langle \mathbf{v}_n, \mathbf{v}_i \rangle$$

$$\implies \langle \mathbf{x}, \mathbf{v}_i \rangle = x_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle.$$

Let V be a vector space with an inner product. Suppose that $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are nonzero vectors that form an orthogonal set. Given $\mathbf{x} \in V$, let

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 + \dots + \frac{\langle \mathbf{x}, \mathbf{v}_k \rangle}{\langle \mathbf{v}_k, \mathbf{v}_k \rangle} \mathbf{v}_k, \quad \mathbf{o} = \mathbf{x} - \mathbf{p}.$$

Let W denote the span of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Theorem (a) $\mathbf{o} \perp \mathbf{w}$ for all $\mathbf{w} \in W$ (denoted $\mathbf{o} \perp W$).

(b) $\|\mathbf{o}\| = \|\mathbf{x} - \mathbf{p}\| = \min_{\mathbf{w} \in W} \|\mathbf{x} - \mathbf{w}\|.$

Thus \mathbf{p} is the **orthogonal projection** of the vector \mathbf{x} on the subspace W . Also, \mathbf{p} is closer to \mathbf{x} than any other vector in W , and $\|\mathbf{o}\| = \text{dist}(\mathbf{x}, \mathbf{p})$ is the **distance** from \mathbf{x} to W .

Orthogonalization

Let V be a vector space with an inner product.

Suppose $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a basis for V . Let

$$\mathbf{v}_1 = \mathbf{x}_1,$$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\langle \mathbf{x}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1,$$

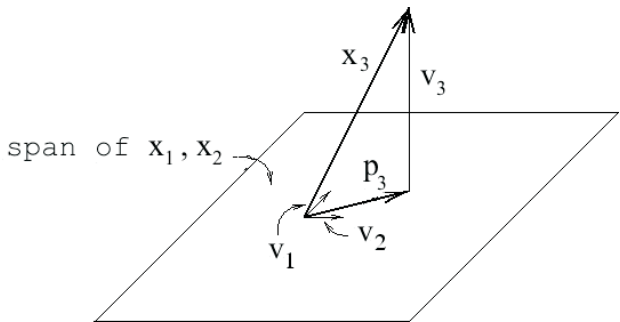
$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\langle \mathbf{x}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{x}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2,$$

.....

$$\mathbf{v}_n = \mathbf{x}_n - \frac{\langle \mathbf{x}_n, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \dots - \frac{\langle \mathbf{x}_n, \mathbf{v}_{n-1} \rangle}{\langle \mathbf{v}_{n-1}, \mathbf{v}_{n-1} \rangle} \mathbf{v}_{n-1}.$$

Then $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V .

The orthogonalization of a basis as described above is called the **Gram-Schmidt process**.



Normalization

Let V be a vector space with an inner product.

Suppose $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is an orthogonal basis for V .

Let $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|}$, \dots , $\mathbf{w}_n = \frac{\mathbf{v}_n}{\|\mathbf{v}_n\|}$.

Then $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ is an orthonormal basis for V .

Theorem Any finite-dimensional vector space with an inner product has an orthonormal basis.

Remark. An infinite-dimensional vector space with an inner product may or may not have an orthonormal basis.