MATH 304

Linear algebra

Lecture 36:
Complex eigenvalues and eigenvectors.
Symmetric and orthogonal matrices.

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

$x =$ real part of z,
$iy =$ imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Examples. • $(1 + i) - (3 + 5i) = (1 - 3) + (i - 5i)$ $= -2 - 4i$;

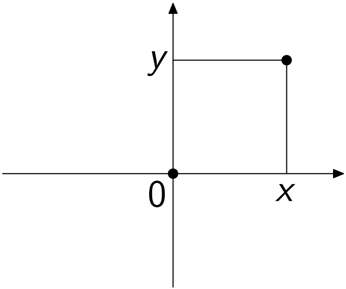
• $(1 + i)(3 + 5i) = 1 \cdot 3 + i \cdot 3 + 1 \cdot 5i + i \cdot 5i$ $= 3 + 3i + 5i + 5i^2 = 3 + 3i + 5i - 5 = -2 + 8i$;

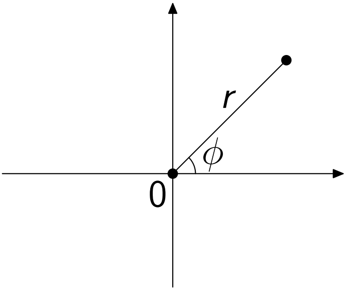
• $(2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13$;

• $i^3 = -i, \;\; i^4 = 1, \;\; i^5 = i.$

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.





$x = r\cos\phi, \quad y = r\sin\phi$

$\implies z = r(\cos\phi + i\sin\phi) = re^{i\phi}.$

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The conjugacy $z \mapsto \bar{z}$ is the reflection of $\mathbb{C}$ in the real line.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Fundamental Theorem of Algebra
Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Complex eigenvalues/eigenvectors

Example. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\det(A - \lambda I) = \lambda^2 + 1$.

Characteristic roots: $\lambda_1 = i$ and $\lambda_2 = -i$.

Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\mathbf{v}_1$, $\mathbf{v}_2$ is a basis of eigenvectors. *In which space?*

Complexification

Instead of the real vector space $\mathbb{R}^2$, we consider a complex vector space $\mathbb{C}^2$ (all complex numbers are admissible as scalars).

The linear operator $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\mathbf{x}) = A\mathbf{x}$ is replaced by the complexified linear operator $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(\mathbf{x}) = A\mathbf{x}$.

The vectors $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$ form a basis for $\mathbb{C}^2$.

Normal matrices

Definition. An $n \times n$ matrix A is called

- **symmetric** if $A^T = A$;
- **orthogonal** if $AA^T = A^T A = I$, i.e., $A^T = A^{-1}$;
- **normal** if $AA^T = A^T A$.

Theorem Let A be an $n \times n$ matrix with real entries. Then
(a) A is normal $\iff$ there exists an orthonormal basis for $\mathbb{C}^n$ consisting of eigenvectors of A;
(b) A is symmetric $\iff$ there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

Example. $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

- A is symmetric.
- A has three eigenvalues: 0, 1, and 2.
- Associated eigenvectors are $\mathbf{v}_1 = (-1, 0, 1)$, $\mathbf{v}_2 = (0, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$, respectively.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1, \mathbf{v}_2, \frac{1}{\sqrt{2}}\mathbf{v}_3$ form an orthonormal basis for $\mathbb{R}^3$.

Theorem Suppose A is a normal matrix. Then for any $\mathbf{x} \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ one has

$$A\mathbf{x} = \lambda\mathbf{x} \iff A^T\mathbf{x} = \overline{\lambda}\mathbf{x}.$$

Thus any normal matrix A shares with A^T all real eigenvalues and the corresponding eigenvectors. Also, $A\mathbf{x} = \lambda\mathbf{x} \iff A\overline{\mathbf{x}} = \overline{\lambda}\,\overline{\mathbf{x}}$ for any matrix A with real entries.

Corollary All eigenvalues λ of a symmetric matrix are real ($\overline{\lambda} = \lambda$). All eigenvalues λ of an orthogonal matrix satisfy $\overline{\lambda} = \lambda^{-1} \iff |\lambda| = 1$.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- A_ϕ is orthogonal
- $\det(A_\phi - \lambda I) = (\cos\phi - \lambda)^2 + \sin^2\phi$.
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$,

$\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.

- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$,

$\mathbf{v}_2 = (1, i)$.

- Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for $\mathbb{C}^2$.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n\times n$ orthogonal matrix.

Theorem There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1, and

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \quad \phi_j \in \mathbb{R}.$$