MATH 304
Linear algebra

Lecture 37:
Rotations in space.
Orthogonal matrices

**Definition.** An \( n \times n \) matrix \( A \) is called **orthogonal** if \( AA^T = A^T A = I \), i.e., \( A^T = A^{-1} \).

**Example.** \( A = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \).

- \( A_\phi \) is orthogonal.
- Eigenvalues: \( \lambda_1 = \cos \phi + i \sin \phi = e^{i\phi} \), \( \lambda_2 = \cos \phi - i \sin \phi = e^{-i\phi} \).
- \( |\lambda_1| = |\lambda_2| = 1 \), \( \overline{\lambda_1} = \lambda_2 \).
- Associated eigenvectors: \( \mathbf{v}_1 = (1, -i) \), \( \mathbf{v}_2 = (1, i) \).
- Vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) form a basis for \( \mathbb{C}^2 \).
Consider a linear operator $L : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $L(x) = Ax$, where $A$ is an $n \times n$ orthogonal matrix.

**Theorem** There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of $L$ relative to this basis has a diagonal block structure

$$
\begin{pmatrix}
D_{\pm 1} & O & \ldots & O \\
O & R_1 & \ldots & O \\
\vdots & \vdots & \ddots & \vdots \\
O & O & \ldots & R_k
\end{pmatrix},
$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or $-1$, and

$$
R_j = \begin{pmatrix}
\cos \phi_j & -\sin \phi_j \\
\sin \phi_j & \cos \phi_j
\end{pmatrix}, \quad \phi_j \in \mathbb{R}.
$$
Theorem  Given an $n \times n$ matrix $A$, the following conditions are equivalent:

(i) $A$ is orthogonal: $A^T = A^{-1}$;
(ii) columns of $A$ form an orthonormal basis for $\mathbb{R}^n$;
(iii) rows of $A$ form an orthonormal basis for $\mathbb{R}^n$.

Proof: Entries of the matrix $A^T A$ are dot products of columns of $A$. Entries of $AA^T$ are dot products of rows of $A$.

Thus an orthogonal matrix is the transition matrix from one orthonormal basis to another.
Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(x) = Ax$, where $A$ is an $n \times n$ matrix.

**Theorem**  The following conditions are equivalent:

(i) $|L(x)| = |x|$ for all $x \in \mathbb{R}^n$;

(ii) $L(x) \cdot L(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$;

(iii) the matrix $A$ is orthogonal.

**Definition.** A transformation $f : \mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** if it preserves distances between points: $|f(x) - f(y)| = |x - y|$.

**Theorem**  Any isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(x) = Ax + x_0$, where $x_0 \in \mathbb{R}^n$ and $A$ is an orthogonal matrix.
Classification of $3 \times 3$ orthogonal matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}.$$ 

$A =$ rotation about a line; $B =$ reflection in a plane; $C =$ rotation about a line combined with reflection in the orthogonal plane.

$$\det A = 1, \quad \det B = \det C = -1.$$ 

$A$ has eigenvalues $1, e^{i\phi}, e^{-i\phi}$. $B$ has eigenvalues $-1, 1, 1$. $C$ has eigenvalues $-1, e^{i\phi}, e^{-i\phi}$. 
Rotations in space

If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).
Clockwise rotations about coordinate axes

\[
\begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \theta & 0 & -\sin \theta \\
0 & 1 & 0 \\
\sin \theta & 0 & \cos \theta
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}
\]
Problem. Find the matrix of the rotation by 90° about the line spanned by the vector \( \mathbf{c} = (1, 2, 2) \). The rotation is assumed to be counterclockwise when looking from the tip of \( \mathbf{c} \).

\[
B = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
is the matrix of (counterclockwise) rotation by 90° about the z-axis.

We need to find an orthonormal basis \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) such that \( \mathbf{v}_3 \) has the same direction as \( \mathbf{c} \). Also, the basis \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) should obey the same hand rule as the standard basis. Then \( B \) is the matrix of the given rotation relative to the basis \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \).
Let $U$ denote the transition matrix from the basis $v_1, v_2, v_3$ to the standard basis (columns of $U$ are vectors $v_1, v_2, v_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $v_1, v_2, v_3$ is going to be an orthonormal basis, the matrix $U$ will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

*Remark.* The basis $v_1, v_2, v_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$. 
**Hint.** Vectors \( \mathbf{a} = (-2, -1, 2), \ \mathbf{b} = (2, -2, 1), \) and \( \mathbf{c} = (1, 2, 2) \) are orthogonal.

We have \( |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3, \) hence \( \mathbf{v}_1 = \frac{1}{3} \mathbf{a}, \ \mathbf{v}_2 = \frac{1}{3} \mathbf{b}, \ \mathbf{v}_3 = \frac{1}{3} \mathbf{c} \) is an orthonormal basis.

Transition matrix: \( \mathbf{U} = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix}. \)

\[
\det \mathbf{U} = \frac{1}{27} \begin{vmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.
\]

(In the case \( \det \mathbf{U} = -1, \) we should interchange vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2. \))
\[
A = UBU^T
\]

\[
= \frac{1}{3} \begin{pmatrix}
-2 & 2 & 1 \\
-1 & -2 & 2 \\
2 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\cdot \frac{1}{3} \begin{pmatrix}
-2 & -1 & 2 \\
2 & -2 & 1 \\
1 & 2 & 2
\end{pmatrix}
\]

\[
= \frac{1}{9} \begin{pmatrix}
2 & 2 & 1 \\
-2 & 1 & 2 \\
1 & -2 & 2
\end{pmatrix}
\begin{pmatrix}
-2 & -1 & 2 \\
2 & -2 & 1 \\
1 & 2 & 2
\end{pmatrix}
\]

\[
= \frac{1}{9} \begin{pmatrix}
1 & -4 & 8 \\
8 & 4 & 1 \\
-4 & 7 & 4
\end{pmatrix}.
\]
\( U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \) is an orthogonal matrix.

\( \det U = 1 \implies U \) is a rotation matrix.

**Problem.** (a) Find the axis of the rotation.  
(b) Find the angle of the rotation.

The axis is the set of points \( \mathbf{x} \in \mathbb{R}^n \) such that \( U\mathbf{x} = \mathbf{x} \iff (U - I)\mathbf{x} = \mathbf{0} \). To find the axis, we apply row reduction to the matrix \( 3(U - I) \):

\[
3U - 3I = \begin{pmatrix} -5 & 2 & 1 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} -3 & 3 & 0 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix}
\]
\[
\begin{pmatrix}
1 & -1 & 0 \\
-1 & -5 & 2 \\
2 & 1 & -1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 0 \\
0 & -6 & 2 \\
2 & 1 & -1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 0 \\
2 & 1 & -1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 0 \\
0 & 3 & -1 \\
0 & 0 & 0 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -1 & 0 \\
0 & 1 & -1/3 \\
0 & 0 & 0 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -1/3 \\
0 & 1 & -1/3 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

Thus \( Ux = x \iff \begin{cases} 
x - z/3 = 0 \\
y - z/3 = 0 
\end{cases} \)

The general solution is \( x = y = t/3, \ z = t, \ t \in \mathbb{R}. \)

\( \implies d = (1, 1, 3) \) is the direction of the axis.
\[ U = \frac{1}{3} \begin{pmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{pmatrix} \]

Let \( \phi \) be the angle of rotation. Then the eigenvalues of \( U \) are 1, \( e^{i\phi} \), and \( e^{-i\phi} \). Therefore

\[ \det(U - \lambda I) = (1 - \lambda)(e^{i\phi} - \lambda)(e^{-i\phi} - \lambda). \]

Besides, \( \det(U - \lambda I) = -\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 \), where \( c_1 = \text{tr} \ U \) (the sum of diagonal entries).

It follows that

\[ \text{tr} \ U = 1 + e^{i\phi} + e^{-i\phi} = 1 + 2 \cos \phi. \]

\[ \text{tr} \ U = -2/3 \implies \cos \phi = -5/6 \implies \phi \approx 146.44^\circ \]