Math 311-503 Spring 2007

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) The planes $x + 2y + 2z = 1$ and $4x + 7y + 4z = 5$ intersect in a line. Find a parametric representation for the line.

Problem 2 (30 pts.) Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 1, 1),\ \mathbf{v}_2 = (1, 2, 2).$$

(i) Find the matrix of the operator L.
(ii) Find the dimensions of the image and the null-space of L.
(iii) Find bases for the image and the null-space of L.

Problem 3 (35 pts.) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 4 (35 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B?
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B?

Problem 5 (30 pts.) Find a quadratic polynomial that is an orthogonal polynomial relative to the inner product

$$\langle p, q \rangle = \int_0^1 x p(x) q(x)\, dx.$$

Bonus Problem 6 (25 pts.) Let S be the set of all points in $\mathbb{R}^3$ that lie at same distance from the planes $x + 2y + 2z = 1$ and $4x + 7y + 4z = 5$. Show that S is the union of two planes and find these planes.

Bonus Problem 7 (35 pts.) (i) Find a matrix exponential $\exp(tC)$, where $C = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$ and $t \in \mathbb{R}$.

(ii) Solve a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y, \\ \dfrac{dy}{dt} = 2y \end{cases}$$

subject to the initial conditions $x(0) = y(0) = 1$.