Math 311-503 Spring 2007

Sample problems for Test 2: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) Let $\mathcal{P}_2$ be the vector space of all polynomials (with real coefficients) of degree at most 2. Determine which of the following subsets of $\mathcal{P}_2$ are vector subspaces. Briefly explain.

(i) The set S_1 of polynomials $p(x) \in \mathcal{P}_2$ such that $p(0) = 0$.

The set S_1 is not empty because it contains the zero polynomial. S_1 is a subspace of $\mathcal{P}_2$ since it is closed under addition and scalar multiplication. Alternatively, S_1 is the null-space of a linear functional $\ell : \mathcal{P}_2 \to \mathbb{R}$ given by $\ell[p(x)] = p(0)$.

(ii) The set S_2 of polynomials $p(x) \in \mathcal{P}_2$ such that $p(0) = 0$ and $p(1) = 0$.

The set S_2 is a subspace of $\mathcal{P}_2$ for the same reason as the set S_1.

(iii) The set S_3 of polynomials $p(x) \in \mathcal{P}_2$ such that $p(0) = 0$ or $p(1) = 0$.

The set S_3 is not a subspace because it is not closed under addition. For example, the polynomials $p_1(x) = x$ and $p_2(x) = x - 1$ belong to S_3 while their sum $p(x) = 2x - 1$ is not in S_3.

(iv) The set S_4 of polynomials $p(x) \in \mathcal{P}_2$ such that $(p(0))^2 + 2(p(1))^2 + (p(2))^2 = 0$.

A polynomial $p(x) \in \mathcal{P}_2$ belongs to S_4 if and only if $p(0) = p(1) = p(2) = 0$. Since the degree of $p(x)$ is at most 2, this means that the set S_4 consists of a single element, the zero polynomial. Thus S_4 is the trivial subspace of $\mathcal{P}_2$.

Problem 2 (20 pts.) Let L be the linear operator on $\mathbb{R}^2$ given by

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of the operator L relative to the basis $\mathbf{v}_1 = (1, 1)$, $\mathbf{v}_2 = (1, -1)$.

Let B denote the matrix of L relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$. The columns of B are coordinates of the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$. We obtain that

$$L(\mathbf{v}_1) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad L(\mathbf{v}_2) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}.$$

Hence $L(\mathbf{v}_1) = \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2$. To determine the coordinates (a, b) of the vector $L(\mathbf{v}_2)$ relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, we need to solve the vector equation $a\mathbf{v}_1 + b\mathbf{v}_2 = L(\mathbf{v}_2)$, which is equivalent to the system

$$\begin{cases} a + b = 3, \\ a - b = -5. \end{cases}$$

Solving the system, we find that $a = -1$, $b = 4$. Thus $L(\mathbf{v}_2) = -\mathbf{v}_1 + 4\mathbf{v}_2$ and

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Alternative solution: Let $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ and let B denote the matrix of L relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$. Let U denote the 2-by-2 matrix whose columns are vectors $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$U = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The matrix U can be used to convert coordinates relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$ into the standard coordinates in $\mathbb{R}^2$. It follows that $B = U^{-1}AU$. Using the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

we obtain that

$$U^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}U.$$

Then

$$B = U^{-1}AU = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 4 \end{pmatrix}.$$

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 5 & 3 & 5 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis for the image of f.

The image of the linear operator f is the subspace of $\mathbb{R}^3$ spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (5, 2, 1)$ and $\mathbf{v}_2 = (3, 1, 0)$ (the third column coincides with the first one). Clearly, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are not parallel. Hence they are linearly independent. Thus $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

(ii) Find a basis for the null-space of f.

The null-space of f is the set of solutions of the vector equation $A\mathbf{x} = \mathbf{0}$. To solve the equation, we shall convert the matrix A to reduced echelon form. Since the right-hand side of the equation is the zero vector, elementary row operations do not change the solution set.

First we interchange the first row of the matrix A with the third one:

$$\begin{pmatrix} 5 & 3 & 5 \\ 2 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 5 & 3 & 5 \end{pmatrix}.$$

Then we subtract 2 times the first row from the second row and 5 times the first row from the third row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 5 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix}.$$

Finally, we subtract 3 times the second row from the third row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that the vector equation $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $x + z = y = 0$, where $\mathbf{x} = (x, y, z)$. The general solution of the system is $x = -t$, $y = 0$, $z = t$ for an arbitrary $t \in \mathbb{R}$. That is, $\mathbf{x} = (-t, 0, t) = t(-1, 0, 1)$, where $t \in \mathbb{R}$. Thus the null-space of the linear operator f is the line $t(-1, 0, 1)$. The vector $(-1, 0, 1)$ is a basis for this line.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. The determinant is easily evaluated using the expansion by the third row:

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((1-\lambda)^2 - 1\big) = -\lambda(1-\lambda)(2-\lambda).$$

Hence the matrix B has three eigenvalues: 0, 1, and 2.

(ii) For each eigenvalue of B, find an associated eigenvector.

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$.

First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + y = 0, \\ z = 0. \end{cases}$$

The general solution is $x = -t$, $y = t$, $z = 0$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (-1, 1, 0)$ is an eigenvector of B associated with the eigenvalue 0.

Next consider the case $\lambda = 1$. We obtain that

$$(B - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + z = 0, \\ y + z = 0. \end{cases}$$

The general solution is $x = -t$, $y = -t$, $z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (-1, -1, 1)$ is an eigenvector of B associated with the eigenvalue 1.

It remains to consider the case $\lambda = 2$. In this case,

$$(B - 2I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - y = 0, \\ z = 0. \end{cases}$$

The general solution is $x = t$, $y = t$, $z = 0$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 0)$ is an eigenvector of B associated with the eigenvalue 2.

(iii) Is there a basis for $\mathbb{R}^3$ consisting of eigenvectors of B?

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, -1, 1)$, and $\mathbf{v}_3 = (1, 1, 0)$ are eigenvectors of the matrix B associated with distinct eigenvalues 0, 1, and 2, respectively. Therefore these vectors are linearly independent. It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

Bonus Problem 5 (25 pts.) Let $f_1, f_2, f_3, \dots$ be the Fibonacci numbers defined by $f_1 = f_2 = 1$, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Find $\lim\limits_{n\to\infty} \dfrac{f_{n+1}}{f_n}$.

For any integer $n \geq 1$ define a two-dimensional vector $\mathbf{v}_n = (f_{n+1}, f_n)$. It follows from the definition of the Fibonacci numbers that

$$\begin{pmatrix} f_{n+2} \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}.$$

That is, $\mathbf{v}_{n+1} = A\mathbf{v}_n$ for $n = 1, 2, \dots$, where

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

In particular, $\mathbf{v}_2 = A\mathbf{v}_1$, $\mathbf{v}_3 = A\mathbf{v}_2 = A^2\mathbf{v}_1$, $\mathbf{v}_4 = A\mathbf{v}_3 = A^3\mathbf{v}_1$, and so on. In general, $\mathbf{v}_n = A^{n-1}\mathbf{v}_1$ for $n = 2, 3, \dots$.

The characteristic equation for the matrix A is $\lambda^2 - \lambda - 1 = 0$. This equation has two roots $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$. It is easy to see that $\lambda_1 > 1$ and $-1 < \lambda_2 < 0$. Let $\mathbf{w}_1 = (x_1, y_1)$ and $\mathbf{w}_2 = (x_2, y_2)$ be eigenvectors of A associated with the eigenvalues λ_1 and λ_2, respectively. The vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ are linearly independent, hence they form a basis for $\mathbb{R}^2$. In particular, the vector $\mathbf{v}_1 = (1, 1)$ is represented as $a\mathbf{w}_1 + b\mathbf{w}_2$ for some $a, b \in \mathbb{R}$. Note that $a \neq 0$ and $b \neq 0$ because $\mathbf{v}_1$ is not an eigenvector of A.

Since $\mathbf{v}_1 = a\mathbf{w}_1 + b\mathbf{w}_2$, it follows that

$$\mathbf{v}_n = A^{n-1}\mathbf{v}_1 = A^{n-1}(a\mathbf{w}_1 + b\mathbf{w}_2) = aA^{n-1}\mathbf{w}_1 + bA^{n-1}\mathbf{w}_2 = a\lambda_1^{n-1}\mathbf{w}_1 + b\lambda_2^{n-1}\mathbf{w}_2$$

for $n \geq 2$. Then $f_n = a\lambda_1^{n-1}y_1 + b\lambda_2^{n-1}y_2$. Therefore

$$\frac{f_{n+1}}{f_n} = \frac{a\lambda_1^n y_1 + b\lambda_2^n y_2}{a\lambda_1^{n-1} y_1 + b\lambda_2^{n-1} y_2} = \lambda_1 \frac{ay_1 + b(\lambda_2/\lambda_1)^n y_2}{ay_1 + b(\lambda_2/\lambda_1)^{n-1} y_2}.$$

Observe that $y_1 \neq 0$ because $(1, 0)$ is not an eigenvector of A. Besides, $|\lambda_2/\lambda_1| < 1$ since $\lambda_1 > 1$ and $|\lambda_2| < 1$. Thus

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \lambda_1 = \frac{1+\sqrt{5}}{2}.$$