MATH 311
Topics in Applied Mathematics

Lecture 21:
Boundary value problems.
Separation of variables.
Differential equations

A **differential equation** is an equation involving an unknown function and certain of its derivatives.

An **ordinary differential equation (ODE)** is an equation involving an unknown function of one variable and certain of its derivatives.

A **partial differential equation (PDE)** is an equation involving an unknown function of two or more variables and certain of its partial derivatives.
Examples

\[ x^2 + 2x + 1 = 0 \]  \hspace{1cm} \text{(algebraic equation)}

\[ f(2x) = 2(f(x))^2 - 1 \]  \hspace{1cm} \text{(functional equation)}

\[ f'(t) + t^2 f(t) = 4 \]  \hspace{1cm} \text{(ODE)}

\[ \frac{\partial u}{\partial x} + 3 \frac{\partial^2 u}{\partial x \partial y} - u \frac{\partial u}{\partial y} \]  \hspace{1cm} \text{(not an equation)}

\[ \frac{\partial u}{\partial x} - 5 \frac{\partial u}{\partial y} = u \]  \hspace{1cm} \text{(PDE)}

\[ u + u^2 = \frac{\partial^2 u}{\partial x \partial y}(0, 0) \]  \hspace{1cm} \text{(functional-differential equation)}
heat equation: \[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

wave equation: \[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Laplace's equation: \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

In the first two equations, \( u = u(x, t) \). In the latter one, \( u = u(x, y) \).
heat equation: \[ \frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

wave equation: \[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]

Laplace’s equation: \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \]

In the first two equations, \( u = u(x, y, t) \). In the latter one, \( u = u(x, y, z) \).
Initial and boundary conditions for ODEs

\[ y'(t) = y(t), \quad 0 \leq t \leq L. \]

General solution: \( y(t) = C_1 e^t, \) where \( C_1 = \text{const.} \)

To determine a unique solution, we need one initial condition.

For example, \( y(0) = 1. \) Then \( y(t) = e^t \) is the unique solution.
\[ y''(t) = -y(t), \ 0 \leq t \leq L. \]

General solution: \[ y(t) = C_1 \cos t + C_2 \sin t, \] where \( C_1, C_2 \) are constant.

To determine a unique solution, we need two initial conditions. For example, \( y(0) = 1, y'(0) = 0. \) Then \( y(t) = \cos t \) is the unique solution.

Alternatively, we may impose boundary conditions. For example, \( y(0) = 0, y(L) = 1. \) In the case \( L = \pi/2, y(t) = \sin t \) is the unique solution.
PDE

\[
\frac{\partial^2 u}{\partial w \partial z} = 0, \quad u = u(w, z)
\]

Domain: \(a_1 \leq w \leq a_2, \ b_1 \leq z \leq b_2\).
(we allow intervals \([a_1, a_2]\) and \([b_1, b_2]\) to be infinite or semi-infinite)

\[
\frac{\partial}{\partial w} \left( \frac{\partial u}{\partial z} \right) = 0, \quad \frac{\partial u}{\partial z}(w, z) = \gamma(z)
\]

\[
\begin{align*}
\int_{z_0}^{z} \gamma(\xi) \, d\xi + C(w) \\
u(w, z) = B(z) + C(w)
\end{align*}
\]

( general solution)
Wave equation

\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \]

Change of independent variables:

\[ w = x + ct, \quad z = x - ct. \]

How does the equation look in new coordinates?

\[ \frac{\partial}{\partial t} = \frac{\partial w}{\partial t} \frac{\partial}{\partial w} + \frac{\partial z}{\partial t} \frac{\partial}{\partial z} = c \frac{\partial}{\partial w} - c \frac{\partial}{\partial z} \]

\[ \frac{\partial}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial}{\partial w} + \frac{\partial z}{\partial x} \frac{\partial}{\partial z} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z} \]
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \left( \frac{\partial}{\partial w} - \frac{\partial}{\partial z} \right) \left( \frac{\partial}{\partial w} - \frac{\partial}{\partial z} \right) u \]

\[ = c^2 \left( \frac{\partial^2 u}{\partial w^2} - 2 \frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2} \right). \]

\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2 \frac{\partial^2 u}{\partial w \partial z} + \frac{\partial^2 u}{\partial z^2}. \]

\[ \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = -4c^2 \frac{\partial^2 u}{\partial w \partial z}. \]

Wave equation in new coordinates: \( \frac{\partial^2 u}{\partial w \partial z} = 0. \)

General solution: \( u(x, t) = B(x - ct) + C(x + ct) \)

(d’Alembert, 1747)
Boundary conditions for PDEs

Heat equation: \( \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L, \quad 0 \leq t \leq T. \)

Initial condition: \( u(x, 0) = f(x), \) where \( f : [0, L] \rightarrow \mathbb{R}. \)

Boundary conditions: \( u(0, t) = u_1(t), \) \( u(L, t) = u_2(t), \) where \( u_1, u_2 : [0, T] \rightarrow \mathbb{R}. \)

Boundary conditions of the first kind: prescribed temperature.
Another boundary conditions: \( \frac{\partial u}{\partial x}(0, t) = \phi_1(t), \)
\( \frac{\partial u}{\partial x}(L, t) = \phi_2(t), \) where \( \phi_1, \phi_2 : [0, T] \rightarrow \mathbb{R}. \)

Boundary conditions of the **second kind**: prescribed heat flux.

A particular case: \( \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0 \)
( insulated boundary ).
Robin conditions:

\[-\frac{\partial u}{\partial x}(0, t) = -h \cdot \left( u(0, t) - u_1(t) \right),\]

\[-\frac{\partial u}{\partial x}(L, t) = h \cdot \left( u(L, t) - u_2(t) \right),\]

where \( h = \text{const} > 0 \) and \( u_1, u_2 : [0, T] \to \mathbb{R} \).

Boundary conditions of the **third kind**: Newton’s law of cooling.

Also, we may consider **mixed** boundary conditions, for example, \( u(0, t) = u_1(t), \frac{\partial u}{\partial x}(L, t) = \phi_2(t) \).
Wave equation

\[
\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L, \; 0 \leq t \leq T.
\]

Two initial conditions: \( u(x, 0) = f(x), \)
\( \frac{\partial u}{\partial t}(x, 0) = g(x), \) where \( f, g : [0, L] \rightarrow \mathbb{R}. \)

Some boundary conditions: \( u(0, t) = u(L, t) = 0. \)

Dirichlet conditions: fixed ends.

Another boundary conditions:
\( \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0. \)

Neumann conditions: free ends.
Linear equations

An equation is called **linear** if it can be written in the form

$$L(u) = f,$$

where $L : V_1 \to V_2$ is a linear map, $f \in V_2$ is given, and $u \in V_1$ is the unknown. If $f = 0$ then the linear equation is called **homogeneous**.

**Theorem** The general solution of a linear equation $L(u) = f$ is

$$u = u_1 + u_0,$$

where $u_1$ is a particular solution and $u_0$ is the general solution of the homogeneous equation $L(u) = 0$. 
Linear differential operators

- ordinary differential operator:
  \[ L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2 \] (\(g_0, g_1, g_2\) are functions)

- heat operator: \( L = \frac{\partial}{\partial t} - k \frac{\partial^2}{\partial x^2} \)

- wave operator: \( L = \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \)
  (a.k.a. the d’Alembertian; denoted by \(\Box\)).

- Laplace’s operator: \( L = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \)
  (a.k.a. the Laplacian; denoted by \(\Delta\) or \(\nabla^2\)).
How do we solve a linear homogeneous PDE?

*Step 1:* Find some solutions.

*Step 2:* Form linear combinations of solutions obtained on Step 1.

*Step 3:* Show that every solution can be approximated by solutions obtained on Step 2.

Similarly, we solve a linear homogeneous PDE with linear homogeneous boundary conditions (boundary problem).

One way to complete Step 1: the method of separation of variables.
Separation of variables

The method applies to certain linear PDEs. It is used to find some solutions.

**Basic idea:** to find a solution of the PDE (function of many variables) as a combination of several functions, each depending only on one variable.

For example, \( u(x, t) = B(x) + C(t) \) or \( u(x, t) = B(x)C(t) \).

The first example works perfectly for one equation: \( \frac{\partial^2 u}{\partial t \partial x} = 0 \).

The second example proved useful for **many** equations.
Heat equation

\[ \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \]

Suppose \( u(x, t) = \phi(x)G(t) \). Then

\[ \frac{\partial u}{\partial t} = \phi(x) \frac{dG}{dt}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 \phi}{dx^2} G(t). \]

Hence

\[ \phi(x) \frac{dG}{dt} = k \frac{d^2 \phi}{dx^2} G(t). \]

Divide both sides by \( k \cdot \phi(x) \cdot G(t) = k \cdot u(x, t) \):

\[ \frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2 \phi}{dx^2}. \]
It follows that
\[ \frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda = \text{const.} \]

\( \lambda \) is called the \textbf{separation constant}. The variables have been separated:
\[ \frac{d^2\phi}{dx^2} = -\lambda \phi, \]
\[ \frac{dG}{dt} = -\lambda kG. \]

**Proposition**  Suppose \( \phi \) and \( G \) are solutions of the above ODEs for the same value of \( \lambda \). Then  
\( u(x, t) = \phi(x)G(t) \) is a solution of the heat equation.

**Example.**  \( u(x, t) = e^{-kt} \sin x. \)
\[ \frac{dG}{dt} = -\lambda kG \]

General solution: \( G(t) = C_0 e^{-\lambda kt}, \) \( C_0 = \text{const.} \)

\[ \frac{d^2 \phi}{dx^2} = -\lambda \phi \]

Three cases: \( \lambda > 0, \lambda = 0, \lambda < 0. \)

Case 1: \( \lambda > 0. \) Then \( \lambda = \mu^2, \) where \( \mu > 0. \)
\( \phi(x) = C_1 \cos \mu x + C_2 \sin \mu x, \quad C_1, C_2 = \text{const.} \)

Case 2: \( \lambda = 0. \) \( \phi(x) = C_1 + C_2 x. \)

Case 3: \( \lambda < 0. \) Then \( \lambda = -\mu^2, \) where \( \mu > 0. \)
\( \phi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}. \)
**Theorem**  For any $C_1, C_2 \in \mathbb{R}$ and $\mu > 0$, the functions

$$u_+(x, t) = e^{-k\mu^2 t}(C_1 \cos \mu x + C_2 \sin \mu x),$$
$$u_0(x, t) = C_1 + C_2 x,$$
$$u_-(x, t) = e^{k\mu^2 t}(C_1 e^{\mu x} + C_2 e^{-\mu x})$$

are solutions of the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$
Laplace’s equation inside a rectangle

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < L, \ 0 < y < H)
\]

Boundary conditions:

\[
\begin{align*}
  u(0, y) &= g_1(y) \\
  u(L, y) &= g_2(y) \\
  u(x, 0) &= f_1(x) \\
  u(x, H) &= f_2(x)
\end{align*}
\]
Principle of superposition:

\[ u = u_1 + u_2 + u_3 + u_4, \]

where

\[ \nabla^2 u_1 = \nabla^2 u_2 = \nabla^2 u_3 = \nabla^2 u_4 = 0, \]

\[ u_1(x, 0) = f_1(x), \quad u_1(0, y) = u_1(L, y) = u_1(x, H) = 0; \]
\[ u_2(L, y) = g_2(y), \quad u_2(0, y) = u_2(x, 0) = u_2(x, H) = 0; \]
\[ u_3(x, H) = f_2(x), \quad u_3(0, y) = u_3(L, y) = u_3(x, 0) = 0; \]
\[ u_4(0, y) = g_1(y), \quad u_4(L, y) = u_4(x, 0) = u_4(x, H) = 0. \]
Reduced boundary value problem

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (0 < x < L, \ 0 < y < H) \]

Boundary conditions:

\[ u(0, y) = 0 \]
\[ u(L, y) = 0 \]
\[ u(x, 0) = f_1(x) \]
\[ u(x, H) = 0 \]
Separation of variables

We are looking for a solution \( u(x, y) = \phi(x)h(y) \) satisfying all 3 homogeneous boundary conditions. (Next step will be to combine such solutions into one that satisfies the nonhomogeneous boundary condition as well.)

PDE holds if

\[
\frac{d^2\phi}{dx^2} = -\lambda \phi \quad \text{and} \quad \frac{d^2h}{dy^2} = \lambda h
\]

for the same constant \( \lambda \).

Boundary conditions \( u(0, y) = u(L, y) = 0 \) hold if

\[
\phi(0) = \phi(L) = 0.
\]

Boundary condition \( u(x, H) = 0 \) holds if

\[
h(H) = 0.
\]
Eigenvalue problem: $\phi'' = -\lambda \phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin \frac{n\pi x}{L}$.

Dependence on $y$:

$h'' = \lambda h$, $h(H) = 0$.

$\implies h(y) = C_0 \sinh \sqrt{\lambda} (y - H)$

Solution of Laplace’s equation:

$u(x, y) = \sin \frac{n\pi x}{L} \sinh \frac{n\pi (y - H)}{L}$, $n = 1, 2, \ldots$