MATH 311

Topics in Applied Mathematics I

Lecture 15:

Basis and dimension.

Spanning set

Let S be a subset of a vector space V.

Definition. The **span** of the set S is the smallest subspace $W \subset V$ that contains S. If S is not empty then $W = \operatorname{Span}(S)$ consists of all linear combinations $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k$ such that $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S$ and $r_1, \ldots, r_k \in \mathbb{R}$.

We say that the set S spans the subspace W or that S is a spanning set for W.

Remarks. • If S_1 is a spanning set for a vector space V and $S_1 \subset S_2 \subset V$, then S_2 is also a spanning set for V.

• If $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_k$ is a spanning set for V and \mathbf{v}_0 is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ then $\mathbf{v}_1, \dots, \mathbf{v}_k$ is also a spanning set for V.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k=\mathbf{0},$$

where the coefficients $r_1, \ldots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k=\mathbf{0} \implies r_1=\cdots=r_k=0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Theorem Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$ are linearly dependent if and only if one of them is a linear combination of the other k-1 vectors.

Basis

Definition. Let V be a vector space. Any linearly independent spanning set for V is called a **basis**.

Suppose that a set $S \subset V$ is a basis for V.

"Spanning set" means that any vector $\mathbf{v} \in V$ can be represented as a linear combination

$$\mathbf{v} = r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \cdots + r_k \mathbf{v}_k,$$

where $\mathbf{v}_1, \dots, \mathbf{v}_k$ are distinct vectors from S and $r_1, \dots, r_k \in \mathbb{R}$. "Linearly independent" implies that the above representation is unique:

$$\mathbf{v} = r_1 \mathbf{v}_1 + r_2 \mathbf{v}_2 + \dots + r_k \mathbf{v}_k = r'_1 \mathbf{v}_1 + r'_2 \mathbf{v}_2 + \dots + r'_k \mathbf{v}_k$$

$$\implies (r_1 - r'_1) \mathbf{v}_1 + (r_2 - r'_2) \mathbf{v}_2 + \dots + (r_k - r'_k) \mathbf{v}_k = \mathbf{0}$$

$$\implies r_1 - r'_1 = r_2 - r'_2 = \dots = r_k - r'_k = \mathbf{0}$$

Examples. • Standard basis for \mathbb{R}^n : $\mathbf{e}_1 = (1, 0, 0, \dots, 0, 0), \ \mathbf{e}_2 = (0, 1, 0, \dots, 0, 0), \dots$

$$\mathbf{e}_n = (0, 0, 0, \dots, 0, 1).$$

Indeed, $(x_1, x_2, \dots, x_n) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n.$

• Matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

form a basis for
$$\mathcal{M}_{2,2}(\mathbb{R})$$
.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

• Polynomials 1 x
$$x^2$$
 x^{n-1} form a basis for

- Polynomials $1, x, x^2, \dots, x^{n-1}$ form a basis for $\mathcal{P}_n = \{a_0 + a_1x + \dots + a_{n-1}x^{n-1} : a_i \in \mathbb{R}\}.$
- The infinite set $\{1, x, x^2, \dots, x^n, \dots\}$ is a basis for \mathcal{P} , the space of all polynomials.

Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$.

The vector equation $r_1\mathbf{v}_1+r_2\mathbf{v}_2+\cdots+r_k\mathbf{v}_k=\mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x}=\mathbf{v}$, where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

$$r_1\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} + r_2\begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{pmatrix} + \cdots + r_k\begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \iff$$

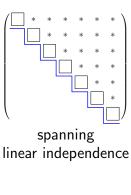
$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \iff A\mathbf{x} = \mathbf{v}$$

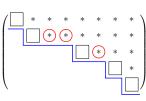
Let $\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$ and $r_1, r_2, \dots, r_k \in \mathbb{R}$. The vector equation $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k = \mathbf{v}$ is equivalent to the matrix equation $A\mathbf{x} = \mathbf{v}$, where

$$A = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k), \qquad \mathbf{x} = \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

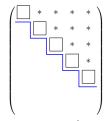
That is, A is the $n \times k$ matrix such that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are consecutive columns of A.

- Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ span \mathbb{R}^n if the row echelon form of A has no zero rows.
- Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly independent if the row echelon form of A has a leading entry in each column (no free variables).

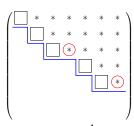




spanning no linear independence



no spanning linear independence



no spanning no linear independence

Bases for \mathbb{R}^n

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ be vectors in \mathbb{R}^n .

Theorem 1 If k < n then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ do not span \mathbb{R}^n .

Theorem 2 If k > n then the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are linearly dependent.

Theorem 3 If k = n then the following conditions are equivalent:

- (i) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for \mathbb{R}^n ;
- (ii) $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a spanning set for \mathbb{R}^n ;
- (iii) $\{v_1, v_2, \dots, v_n\}$ is a linearly independent set.

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in \mathbb{R}^3 .

Vectors \mathbf{v}_1 and \mathbf{v}_2 are linearly independent (as they are not parallel), but they do not span \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent since

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -(-2) = 2 \neq 0.$$

Therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for \mathbb{R}^3 .

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ span \mathbb{R}^3 (because $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ already span \mathbb{R}^3), but they are linearly dependent.

Dimension

Theorem 1 Any vector space has a basis.

Theorem 2 If a vector space V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted dim V, is the number of elements in any of its bases.

Examples. • dim $\mathbb{R}^n = n$

• $\mathcal{M}_{2,2}(\mathbb{R})$: the space of 2×2 matrices dim $\mathcal{M}_{2,2}(\mathbb{R})=4$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices $\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

- \mathcal{P}_n : polynomials of degree less than n dim $\mathcal{P}_n = n$
- \bullet $\ensuremath{\mathcal{P}}$: the space of all polynomials $\dim \ensuremath{\mathcal{P}} = \infty$
- $\bullet \ \ \{ {\bm 0} \} \text{: the trivial vector space} \\ \dim \left\{ {\bm 0} \right\} = 0$

Problem. Find the dimension of the plane x + 2z = 0 in \mathbb{R}^3 .

The general solution of the equation x+2z=0 is $\begin{cases} x=-2s \\ y=t \\ z=s \end{cases}$ $(t,s\in\mathbb{R})$

That is, (x, y, z) = (-2s, t, s) = t(0, 1, 0) + s(-2, 0, 1). Hence the plane is the span of vectors $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (-2, 0, 1)$. These vectors are linearly independent as they are not parallel.

Thus $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis so that the dimension of the plane is 2.

How to find a basis?

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

- (i) S is a linearly independent spanning set for V, i.e., a basis;
- (ii) S is a minimal spanning set for V;
- (iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

Theorem Let V be a vector space. Then **(i)** any spanning set for V can be reduced to a minimal spanning set;

(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

Corollary 1 Any spanning set contains a basis while any linearly independent set is contained in a basis.

Corollary 2 A vector space is finite-dimensional if and only if it is spanned by a finite set.