Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

The intersection point (x, y, z) is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

To solve the system, we convert its augmented matrix into reduced row echelon form using elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & -3 & 0 & -5 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 0 & -3 & 3 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -2 \\ 0 & -5 & 1 & -6 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & -5 & 1 & -6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & -4 & -\frac{8}{3} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right).$$

Thus the three planes intersect at the point $(-1, \frac{4}{3}, \frac{2}{3})$.

Alternative solution: The intersection point (x, y, z) is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

Adding all three equations, we obtain $4x = -4$. Hence $x = -1$. Substituting $x = -1$ into the second equation, we obtain $y = \frac{4}{3}$. Substituting $x = -1$ and $y = \frac{4}{3}$ into the third equation, we obtain $z = \frac{2}{3}$. It is easy to check that $x = -1$, $y = \frac{4}{3}$, $z = \frac{2}{3}$ is indeed a solution of the system. Thus $(-1, \frac{4}{3}, \frac{2}{3})$ is the unique intersection point.

Problem 2 Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 1, 1), \ \mathbf{v}_2 = (1, 2, 2).$$

(i) Find the matrix of the operator L.

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{v}\cdot\mathbf{v}_1 = x+y+z$ and $L(\mathbf{v}) = (x+y+z, 2(x+y+z), 2(x+y+z))$. Let A denote the matrix of the linear operator L. The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$. Therefore

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Alternative solution: Given a vector $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, let $\alpha = \mathbf{v} \cdot \mathbf{v}_1$ and $(x_1, y_1, z_1) = L(\mathbf{v})$. In terms of matrix algebra, we have

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} (\alpha) = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(note that scalar multiplication of a column vector is equivalent to multiplication by a 1×1 matrix but the matrix has to be on the right as otherwise the matrix product is not defined). It follows that the matrix of the operator L is

$$\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

(ii) Find the dimensions of the range and the kernel of L.

The range $\text{Range}(L)$ of the linear operator L is the subspace of all vectors of the form $L(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^3$. It is easy to see that $\text{Range}(L)$ is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$. Hence $\dim \text{Range}(L) = 1$.

The kernel $\ker(L)$ of the operator L is the subspace of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Clearly, $L(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} \cdot \mathbf{v}_1 = 0$. Therefore $\ker(L)$ is the plane $x + y + z = 0$ orthogonal to $\mathbf{v}_1$ and passing through the origin. Its dimension is 2.

(iii) Find bases for the range and the kernel of L.

Since the range of L is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$, this vector is a basis for the range. The kernel of L is the plane given by the equation $x + y + z = 0$. The general solution of the equation is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. It gives rise to a parametric representation $t(-1, 1, 0) + s(-1, 0, 1)$ of the plane. Thus the kernel of L is spanned by the vectors $(-1, 1, 0)$ and $(-1, 0, 1)$. Since the two vectors are linearly independent, they form a basis for $\ker(L)$.

Problem 3 Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

Let U be a 3×3 matrix such that its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the determinant of U, we subtract the second row from the first one and then expand by the first row:

$$\det U = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Since $\det U \neq 0$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. It follows that they form a basis for $\mathbb{R}^3$.

(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let A denote the matrix of L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By definition, the columns of A are coordinates of vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since $L(\mathbf{v}_1) = \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$, $L(\mathbf{v}_2) = \mathbf{v}_3 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$, we obtain

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(iii) Find the matrix of the operator L relative to the standard basis.

Let S denote the matrix of L relative to the standard basis for $\mathbb{R}^3$. We have $S = UAU^{-1}$, where A is the matrix of L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (already found) and U is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are consecutive columns of U):

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the inverse U^{-1}, we merge the matrix U with the identity matrix I into one 3×6 matrix and apply row reduction to convert the left half U of this matrix into I. Simultaneously, the right half I will be converted into U^{-1}:

$$(U|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = (I|U^{-1}).$$

Thus

$$S = UAU^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Alternative solution: Let S denote the matrix of L relative to the standard basis $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$. By definition, the columns of S are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$. It is easy to observe that $\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$, $\mathbf{e}_3 = \mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{e}_2 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Therefore

$$\begin{aligned} L(\mathbf{e}_1) &= L(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = -L(\mathbf{v}_1) + L(\mathbf{v}_2) + L(\mathbf{v}_3) = -\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_1 = (1,0,2), \\ L(\mathbf{e}_2) &= L(\mathbf{v}_1 - \mathbf{v}_3) = L(\mathbf{v}_1) - L(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1 = (0,0,-1), \\ L(\mathbf{e}_3) &= L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{v}_2 - \mathbf{v}_3 = (0,1,-1). \end{aligned}$$

Thus

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Problem 4 Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 2$$

$$= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1-\lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3-\lambda).$$

Hence the matrix B has two eigenvalues: 0 and 3.

(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently, $\mathbf{x} = t(-1,1,0) + s(-1,0,1)$. Hence the eigenspace of B associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors $\mathbf{v}_1 = (-1,1,0)$ and $\mathbf{v}_2 = (-1,0,1)$.

Now consider the case $\lambda = 3$. We obtain that

$$(B - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = y = z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B. They are linearly independent since the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

It is easy to check that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. To transform the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into an orthogonal one, we only need to orthogonalize the pair $\mathbf{v}_1, \mathbf{v}_2$. Using the Gram-Schmidt process, we replace the vector $\mathbf{v}_2$ by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = (-1/2, -1/2, 1).$$

Now $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$ is an orthogonal basis for $\mathbb{R}^3$. Since $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, it is also an eigenvector of B associated with the eigenvalue 0.

Finally, vectors $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$, and $\mathbf{w}_3 = \dfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$ form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B. We get that $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{u}\| = \sqrt{3/2}$, and $\|\mathbf{v}_3\| = \sqrt{3}$. Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1, -1, 2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

Remark. We cannot apply the Gram-Schmidt process to eigenvectors from different eigenspaces since a linear combination of them would not be an eigenvector. Hence for an orthogonal basis of eigenvectors to exist, different eigenspaces must be orthogonal to one another. It turns out that an orthonormal basis of eigenvectors exists if and only if the matrix is symmetric.

(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B associated with eigenvalues 0, 0, and 3, respectively. Since these vectors form a basis for $\mathbb{R}^3$, it follows that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) while D is the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = B\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 5 Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

(i) Find the distance from the point $\mathbf{y} = (0,0,0,4)$ to the subspace V.
(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

The vector $\mathbf{y}$ is uniquely represented as $\mathbf{y} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o}$ is orthogonal to V, that is, $\mathbf{o} \in V^\perp$. The vector $\mathbf{p}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace V. Since $(V^\perp)^\perp = V$, the vector $\mathbf{o}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace $V^\perp$. It follows that the distance from the point $\mathbf{y}$ to V equals $\|\mathbf{o}\|$ while the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{p}\|$.

The orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto the subspace V is easily computed when we have an orthogonal basis for V. To get such a basis, we apply the Gram-Schmidt orthogonalization process to the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1,1,0,0), \qquad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (2,0,-1,1) - \frac{2}{2}(1,1,0,0) = (1,-1,-1,1),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0,1,1,0) - \frac{1}{2}(1,1,0,0) - \frac{-2}{4}(1,-1,-1,1) = (0,0,1/2,1/2).$$

Now that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V we obtain

$$\mathbf{p} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 =$$

$$= \frac{0}{2}(1,1,0,0) + \frac{4}{4}(1,-1,-1,1) + \frac{2}{1/2}(0,0,1/2,1/2) = (1,-1,1,3).$$

Consequently, $\mathbf{o} = \mathbf{y} - \mathbf{p} = (0,0,0,4) - (1,-1,1,3) = (-1,1,-1,1)$. Thus the distance from $\mathbf{y}$ to the subspace V equals $\|\mathbf{o}\| = 2$ and the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{p}\| = \sqrt{12} = 2\sqrt{3}$.

Problem 6 Consider a vector field $\mathbf{F}(x,y,z) = xyz\mathbf{e}_1 + xy\mathbf{e}_2 + x^2\mathbf{e}_3$.
(i) Find $\mathrm{curl}(\mathbf{F})$.

$$\mathrm{curl}(\mathbf{F}) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & xy & x^2 \end{vmatrix} = \left(\frac{\partial(x^2)}{\partial y} - \frac{\partial(xy)}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial(xyz)}{\partial z} - \frac{\partial(x^2)}{\partial x}\right)\mathbf{e}_2$$
$$+ \left(\frac{\partial(xy)}{\partial x} - \frac{\partial(xyz)}{\partial y}\right)\mathbf{e}_3 = (xy - 2x)\mathbf{e}_2 + (y - xz)\mathbf{e}_3.$$

(ii) Find the integral of the vector field $\mathrm{curl}(\mathbf{F})$ along a hemisphere $H = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$. Orient the hemisphere by the normal vector $\mathbf{n} = (0,0,1)$ at the point $(0,0,1)$.

According to Stokes' Theorem,

$$\iint_H \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} = \oint_{\partial H} \mathbf{F} \cdot d\mathbf{s},$$

where the boundary ∂H is oriented consistently with H. The boundary is a circle, $\partial H = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 0\}$. It is parametrized (with the right orientation) by a path $\mathbf{x} : [0, 2\pi] \to \mathbb{R}^3$, $\mathbf{x}(t) = (\cos t, \sin t, 0)$. We have $\mathbf{F}(\mathbf{x}(t)) = (0, \cos t\, \sin t, \cos^2 t)$ and $\mathbf{x}'(t) = (-\sin t, \cos t, 0)$. Therefore

$$\oint_{\partial H} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt = \int_0^{2\pi} \cos^2 t\, \sin t\, dt = -\frac{1}{3}\cos^3 t\,\Big|_{t=0}^{2\pi} = 0.$$

Alternative solution: Let us evaluate the surface integral directly. To parametrize the hemisphere H, we use a map $\mathbf{X} : [0, 2\pi] \times [0, \pi/2] \to \mathbb{R}^3$ given by $\mathbf{X}(\lambda, \phi) = (\cos\lambda\cos\phi,\ \sin\lambda\cos\phi,\ \sin\phi)$. Here the parameters λ and ϕ are the longitude and the latitude of a point on H. Then

$$\iint_{\mathbf{X}} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_0^{\pi/2}\int_0^{2\pi} \operatorname{curl}\mathbf{F}\big(\mathbf{X}(\lambda,\phi)\big)\cdot\Big(\frac{\partial\mathbf{X}}{\partial\lambda}\times\frac{\partial\mathbf{X}}{\partial\phi}\Big)\,d\lambda\,d\phi.$$

We have $\dfrac{\partial\mathbf{X}}{\partial\lambda} = (-\sin\lambda\cos\phi,\ \cos\lambda\cos\phi,\ 0)$ and $\dfrac{\partial\mathbf{X}}{\partial\phi} = (-\cos\lambda\sin\phi,\ -\sin\lambda\sin\phi,\ \cos\phi)$. By the above, $\operatorname{curl}\mathbf{F}(x,y,z) = (0,\ xy-2x,\ y-xz)$, therefore

$$\operatorname{curl}\mathbf{F}\big(\mathbf{X}(\lambda,\phi)\big) = (0,\ \cos\lambda\sin\lambda\cos^2\phi - 2\cos\lambda\cos\phi,\ \sin\lambda\cos\phi - \cos\lambda\cos\phi\sin\phi).$$

Consequently,

$$\operatorname{curl}\mathbf{F}\big(\mathbf{X}(\lambda,\phi)\big)\cdot\Big(\frac{\partial\mathbf{X}}{\partial\lambda}\times\frac{\partial\mathbf{X}}{\partial\phi}\Big) =$$

$$= \begin{vmatrix} 0 & \cos\lambda\sin\lambda\cos^2\phi - 2\cos\lambda\cos\phi & \sin\lambda\cos\phi - \cos\lambda\cos\phi\sin\phi \\ -\sin\lambda\cos\phi & \cos\lambda\cos\phi & 0 \\ -\cos\lambda\sin\phi & -\sin\lambda\sin\phi & \cos\phi \end{vmatrix}$$

$$= (\cos\lambda\sin\lambda\cos^2\phi - 2\cos\lambda\cos\phi)\sin\lambda\cos^2\phi$$
$$+ (\sin\lambda\cos\phi - \cos\lambda\cos\phi\sin\phi)(\sin^2\lambda\cos\phi\sin\phi + \cos^2\lambda\cos\phi\sin\phi)$$
$$= (\cos\lambda\sin\lambda\cos^2\phi - 2\cos\lambda\cos\phi)\sin\lambda\cos^2\phi + (\sin\lambda\cos\phi - \cos\lambda\cos\phi\sin\phi)\cos\phi\sin\phi$$
$$= \cos\lambda\sin^2\lambda\cos^4\phi - 2\cos\lambda\sin\lambda\cos^3\phi + \sin\lambda\cos^2\phi\sin\phi - \cos\lambda\cos^2\phi\sin^2\phi.$$

Observe that

$$\int_0^{2\pi}\cos\lambda\sin^2\lambda\,d\lambda = \int_0^{2\pi} 2\cos\lambda\sin\lambda\,d\lambda = \int_0^{2\pi}\sin\lambda\,d\lambda = \int_0^{2\pi}\cos\lambda\,d\lambda = 0$$

(each of the integrals is easily evaluated via the Fundamental Theorem of Calculus). It follows that

$$\int_0^{2\pi}\operatorname{curl}\mathbf{F}\big(\mathbf{X}(\lambda,\phi)\big)\cdot\Big(\frac{\partial\mathbf{X}}{\partial\lambda}\times\frac{\partial\mathbf{X}}{\partial\phi}\Big)\,d\lambda = 0$$

for any value of ϕ. Using Fubini's Theorem, we conclude that

$$\iint_{\mathbf{X}}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0.$$

The latter integral equals the integral of the vector field $\operatorname{curl}\mathbf{F}$ along the hemisphere H provided that the parametrization $\mathbf{X}$ induces the right orientation on H. Otherwise (when the induced orientation is wrong) we need to change the sign. Either way, the integral equals zero.

For completeness, let us check out the orientation induced by $\mathbf{X}$. It is defined by the following field of normals:

$$\mathbf{N} = \frac{\partial\mathbf{X}}{\partial\lambda}\times\frac{\partial\mathbf{X}}{\partial\phi} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ -\sin\lambda\cos\phi & \cos\lambda\cos\phi & 0 \\ -\cos\lambda\sin\phi & -\sin\lambda\sin\phi & \cos\phi \end{vmatrix} = (\cos\lambda\cos^2\phi,\ \sin\lambda\cos^2\phi,\ \cos\phi\sin\phi).$$

In the parameter domain, the reference point $(0,0,1) \in H$ corresponds to any point (λ, ϕ) with $\phi = \pi/2$. Unfortunately, this is a singular point of the parametrization as the field $\mathbf{N}$ vanishes. Instead, we are going to find the limit of unit normals $\dfrac{\mathbf{N}}{\|\mathbf{N}\|}$ as $\phi \to \pi/2$. If $\phi < \pi/2$ then $\cos\phi > 0$. Hence the vector $\mathbf{N} = (\cos\lambda\cos^2\phi,\ \sin\lambda\cos^2\phi,\ \cos\phi\sin\phi)$ has the same direction as the vector $(\cos\lambda\cos\phi,\ \sin\lambda\cos\phi,\ \sin\phi)$. The latter converges to $(0,0,1)$ as $\phi \to \pi/2$. It follows that $\dfrac{\mathbf{N}}{\|\mathbf{N}\|}$ also converges to $(0,0,1)$ as $\phi \to \pi/2$. We conclude that the orientation induced by $\mathbf{X}$ is the right one.

Problem 7 Find the volume of a parallelepiped bounded by planes $x + 2y - z = -1$, $x + 2y - z = 1$, $x - 3y = -5$, $x - 3y = 0$, $2x + y + z = 0$, and $2x + y + z = 2$.

Let P denote the parallelepiped. The volume of P can be found as a triple integral:

$$\text{Volume}(P) = \iiint_P 1\,dx\,dy\,dz.$$

To evaluate the integral, we are going to change variables. New variables are $u = x+2y-z$, $v = x-3y$, and $w = 2x + y + z$. In these variables the parallelepiped P is given by $-1 \le u \le 1$, $-5 \le v \le 0$, $0 \le w \le 2$. It follows that

$$\text{Volume}(P) = \int_0^2 \int_{-5}^0 \int_{-1}^1 \left| \det \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw.$$

Our change of coordinates is linear,

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & -3 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let U denote the above matrix. The Jacobian matrix $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$ equals U at every point of $\mathbb{R}^3$. Consequently, the Jacobian matrix $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ equals U^{-1} everywhere on $\mathbb{R}^3$. We obtain

$$\det U = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -3 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 3 & 0 \\ 1 & -3 & 0 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ 1 & -3 \end{vmatrix} = -12.$$

Hence $\det(U^{-1}) = (\det U)^{-1} = -1/12$. Then

$$\text{Volume}(P) = \int_0^2 \int_{-5}^0 \int_{-1}^1 \left|\det(U^{-1})\right| du\,dv\,dw = \frac{1}{12} \cdot 2 \cdot 5 \cdot 2 = \frac{5}{3}.$$