

MATH 311

Topics in Applied Mathematics I

**Lecture 5:  
Matrix algebra.**

## Matrices

*Definition.* An **m-by-n matrix** is a rectangular array of numbers that has  $m$  rows and  $n$  columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

*Notation:*  $A = (a_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$  or simply  $A = (a_{ij})$  if the dimensions are known.

An  $n$ -dimensional vector can be represented as a  $1 \times n$  matrix (row vector) or as an  $n \times 1$  matrix (column vector):

$$(x_1, x_2, \dots, x_n)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

An  $m \times n$  matrix  $A = (a_{ij})$  can be regarded as a column of  $n$ -dimensional row vectors or as a row of  $m$ -dimensional column vectors:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}, \quad \mathbf{v}_i = (a_{i1}, a_{i2}, \dots, a_{in})$$

$$A = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n), \quad \mathbf{w}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$$

## Vector algebra

Let  $\mathbf{a} = (a_1, a_2, \dots, a_n)$  and  $\mathbf{b} = (b_1, b_2, \dots, b_n)$  be  $n$ -dimensional vectors, and  $r \in \mathbb{R}$  be a scalar.

*Vector sum:*  $\mathbf{a} + \mathbf{b} = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n)$

*Scalar multiple:*  $r\mathbf{a} = (ra_1, ra_2, \dots, ra_n)$

*Zero vector:*  $\mathbf{0} = (0, 0, \dots, 0)$

*Negative of a vector:*  $-\mathbf{b} = (-b_1, -b_2, \dots, -b_n)$

*Vector difference:*

$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) = (a_1 - b_1, a_2 - b_2, \dots, a_n - b_n)$

Given  $n$ -dimensional vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  and scalars  $r_1, r_2, \dots, r_k$ , the expression

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \dots + r_k\mathbf{v}_k$$

is called a **linear combination** of vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ .

Also, *vector addition* and *scalar multiplication* are called **linear operations**.

## Matrix algebra: linear operations

*Definition.* Let  $A = (a_{ij})$  and  $B = (b_{ij})$  be  $m \times n$  matrices. The **sum**  $A + B$  is defined to be the  $m \times n$  matrix  $C = (c_{ij})$  such that  $c_{ij} = a_{ij} + b_{ij}$  for all indices  $i, j$ .

That is, two matrices with the same dimensions can be added by adding their corresponding entries.

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \\ a_{31} + b_{31} & a_{32} + b_{32} \end{pmatrix}$$

*Definition.* Given an  $m \times n$  matrix  $A = (a_{ij})$  and a number  $r$ , the **scalar multiple**  $rA$  is defined to be the  $m \times n$  matrix  $D = (d_{ij})$  such that  $d_{ij} = ra_{ij}$  for all indices  $i, j$ .

That is, to multiply a matrix by a scalar  $r$ , one multiplies each entry of the matrix by  $r$ .

$$r \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & ra_{13} \\ ra_{21} & ra_{22} & ra_{23} \\ ra_{31} & ra_{32} & ra_{33} \end{pmatrix}$$



The  $m \times n$  **zero matrix** (all entries are zeros) is denoted  $O_{mn}$  or simply  $O$ .

**Negative** of a matrix:  $-A$  is defined as  $(-1)A$ .

Matrix **difference**:  $A - B$  is defined as  $A + (-B)$ .

As far as the *linear operations* (addition and scalar multiplication) are concerned, the  $m \times n$  matrices can be regarded as  $mn$ -dimensional vectors.

## Examples

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

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$$A + B = \begin{pmatrix} 5 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A - B = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$2C = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \quad 3D = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix},$$

$$2C + 3D = \begin{pmatrix} 7 & 3 \\ 0 & 5 \end{pmatrix}, \quad A + D \text{ is not defined.}$$

## Properties of linear operations

$$(A + B) + C = A + (B + C)$$

$$A + B = B + A$$

$$A + O = O + A = A$$

$$A + (-A) = (-A) + A = O$$

$$r(sA) = (rs)A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$1A = A$$

$$0A = O$$

## Dot product

*Definition.* The **dot product** of  $n$ -dimensional vectors  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  is a scalar

$$\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n = \sum_{k=1}^n x_ky_k.$$

The dot product is also called the **scalar product**.

## Matrix multiplication

The product of matrices  $A$  and  $B$  is defined if the number of columns in  $A$  matches the number of rows in  $B$ .

*Definition.* Let  $A = (a_{ik})$  be an  $m \times n$  matrix and  $B = (b_{kj})$  be an  $n \times p$  matrix. The **product**  $AB$  is defined to be the  $m \times p$  matrix  $C = (c_{ij})$  such that

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \text{ for all indices } i, j.$$

That is, matrices are multiplied **row by column**:

$$\begin{pmatrix} * & * & * \\ \boxed{*} & \boxed{*} & \boxed{*} \end{pmatrix} \begin{pmatrix} * & * & \boxed{*} & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * & * \\ * & * & \boxed{*} & * \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left( \begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array} \right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\Rightarrow AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

*Examples.*

$$(x_1, x_2, \dots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left( \sum_{k=1}^n x_k y_k \right),$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \dots, x_n) = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \dots & y_1 x_n \\ y_2 x_1 & y_2 x_2 & \dots & y_2 x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_n x_1 & y_n x_2 & \dots & y_n x_n \end{pmatrix}.$$

*Example.*

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \text{ is not defined}$$



System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Matrix representation of the system:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

## Properties of matrix multiplication:

$$(AB)C = A(BC) \quad (\text{associative law})$$

$$(A + B)C = AC + BC \quad (\text{distributive law \#1})$$

$$C(A + B) = CA + CB \quad (\text{distributive law \#2})$$

$$(rA)B = A(rB) = r(AB)$$

*Any of the above identities holds provided that matrix sums and products are well defined.*

If  $A$  and  $B$  are  $n \times n$  matrices, then both  $AB$  and  $BA$  are well defined  $n \times n$  matrices.

However, in general,  $AB \neq BA$ .

*Example.* Let  $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ .

Then  $AB = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ ,  $BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$ .

If  $AB$  does equal  $BA$ , we say that the matrices  $A$  and  $B$  **commute**.

**Problem.** Let  $A$  and  $B$  be arbitrary  $n \times n$  matrices. Is it true that  $(A - B)(A + B) = A^2 - B^2$ ?

$$\begin{aligned}(A - B)(A + B) &= (A - B)A + (A - B)B \\ &= (AA - BA) + (AB - BB) \\ &= A^2 + AB - BA - B^2.\end{aligned}$$

Hence  $(A - B)(A + B) = A^2 - B^2$  if and only if  $A$  commutes with  $B$ .